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From Maxwell to Helmholtz

Bachelorarbeit

zur Erlangung des akademischen Grades

Bachelor

in der Studienrichtung

Technische Mathematik

Angefertigt am *Institut für Numerische Mathematik*

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Linz, November 2017

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1 Introduction

In modern days, everybody is using an electrical device nearly all the time be it a mobile, a digital watch, an e-reader or something similar. What do these things have in common? Of course, they all obey the laws of electromagnetism, which can be described by mathematical equations. Thus, we can discuss the properties and relations in a mathematical sense. The goal of this thesis is to get from the aforementioned Maxwell-equations to the Helmholtz-equation in vector and scalar form in special electromagnetic regimes. The latter is going to be used in the numerical part of this thesis as model for simulation of time-harmonic problems.
The first part covers the theoretical approach, as discussed in the lecture "Mathematical modeling in engineering", on how to model the Maxwell equations, two different types of formulation, the vector potential and the electric field one. And some useful tools we have to use for the last theoretical part on how to get to the Helmholtz equation.
The second part was covered in a lecture during my exchange year at the NTNU (Norwegian University of Science and Technology) in Trondheim. The topic of this class was called "Numerical solutions using finite element method". The project here was to simulate a wi-fi router, which sends out a signal, and to see how the wave disperses in a "real" flat or in empty space or pretty much anywhere one wanted to test the solution.

2 Maxwell's Equations

2.1 Notations, definitions, and some preliminary results

Definition 2.1. *(Curl) Let $V \subset \mathbb{R}^3$ be open and $f : V \longrightarrow \mathbb{R}^3$ be differentiable. Then $curl f := (\frac{\partial f_3(x)}{\partial x_2} - \frac{\partial f_2(x)}{\partial x_3}, \frac{\partial f_1(x)}{\partial x_3} - \frac{\partial f_3(x)}{\partial x_1}, \frac{\partial f_2(x)}{\partial x_1} - \frac{\partial f_1(x)}{\partial x_2})^T$*
We can also write $curl f$ as $\nabla \times f$.

Definition 2.2. *(Divergence) Let $V \subset \mathbb{R}^3$ be open and $f : V \longrightarrow \mathbb{R}^3$ be differentiable. Then $div f := \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}$*
We can also write $div f$ as $\nabla \cdot f$.

Definition 2.3. *(Laplacian) Let $V \subset \mathbb{R}^3$ be open and $f : V \longrightarrow \mathbb{R}^3$ be two times differentiable. Then $\Delta f := \nabla \cdot (\nabla f) = \frac{\partial^2 f_1}{\partial x_1^2} + \frac{\partial^2 f_2}{\partial x_2^2} + \frac{\partial^2 f_3}{\partial x_3^2}$*

Theorem 2.1. *(Stokes' Theorem) Let S be an area in $\mathbb{R}^3$, for which a parameter area $K \subset \mathbb{R}^2$ exists. The parametrization $\varphi : M \longrightarrow \mathbb{R}^3; K \subset M$ open be twice differentiable. Let C be the positiv oriented boundary curve of K with increment wise twice differentiable parametrization $\gamma : [a, b] \longrightarrow \mathbb{R}^2$. ∂S is the curve in $\mathbb{R}^3$ which has the parametrization $\gamma \circ \varphi$. Let f be a continuous differentiable vector field on $\varphi(K)$, then*

$$\int_S curl f \, dS = \int_{\partial S} f \cdot ds \tag{1}$$

Proof. See Theorem 8.50 in [1] ■

Theorem 2.2. *(Gauss' Theorem) Let $V \subset \mathbb{R}^3$ a bounded domain with a sufficiently smooth boundary ∂V. Let M such that $V \subset M$ be open and $f \in C^1(M, \mathbb{R}^3)$. Then*

$$\int_{\partial V} f \cdot n \, dS = \int_V \nabla \cdot f \, dx \tag{2}$$

Proof. See Theorem 8.58 in [1] ■

Lemma 2.1. *Let $f \in C(\Omega)$. If*

$$\int_V f(x) \, dx = 0 \quad \forall \, V \subset \Omega,$$

then

$$f(x) = 0 \, , \forall x \in \Omega$$

Proof. (Indirect proof) Assume $f(x) \neq 0$ for some $x^* \in \Omega$. We know that f is continuous. Therefore, there exists an area around x^*, which we call $U(x^*)$, where $f(x) \neq 0 \, \forall x \in U(x^*)$. Choosing $V = U_{x^*}$, we get that

$$\int_V f(x) \, dx > 0$$

which leads to a contradiction. This yields that $f(x) = 0 \, \forall x \in \Omega$ ■

Lemma 2.2. *For a twice continuously differentiable function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$, and a twice differentiable vector field $F : \mathbb{R}^3 \longrightarrow \mathbb{R}$, the following equations hold.*

$$curl \, curl F = \nabla(\nabla \cdot F) - \Delta F, \tag{3}$$

$$curl(\nabla f) = 0, \tag{4}$$

$$\nabla \cdot (curl F) = 0. \tag{5}$$

Proof. (3):

$$curl \, curl F = curl \begin{pmatrix} \partial_2 F_3 - \partial_3 F_2 \\ \partial_3 F_1 - \partial_1 F_3 \\ \partial_1 F_2 - \partial_2 F_1 \end{pmatrix} = \begin{pmatrix} \partial_2(\partial_1 F_2 - \partial_2 F_1) - \partial_3(\partial_3 F_1 - \partial_1 F_3) \\ \partial_3(\partial_2 F_3 - \partial_3 F_2) - \partial_1(\partial_1 F_2 - \partial_2 F_1) \\ \partial_1(\partial_3 F_1 - \partial_1 F_3) - \partial_2(\partial_2 F_3 - \partial_3 F_2) \end{pmatrix}$$

Now splitting it up and adding some terms we get the right hand side

$$\begin{pmatrix} \partial_1(\partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3) \\ \partial_2(\partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3) \\ \partial_3(\partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3 \end{pmatrix} - \begin{pmatrix} \partial_1^2 F_1 + \partial_2^2 F_1 + \partial_3^2 F_1 \\ \partial_1^2 F_2 + \partial_2^2 F_2 + \partial_3^2 F_2 \\ \partial_1^2 F_3 + \partial_2^2 F_3 + \partial_3^2 F_3 \end{pmatrix} = \nabla(\nabla \cdot F) - \Delta F,$$

(4):

$$curl(\nabla f) = curl \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \end{pmatrix} = \begin{pmatrix} \partial_2 \partial_3 f - \partial_3 \partial_2 f \\ \partial_3 \partial_1 f - \partial_1 \partial_3 f \\ \partial_1 \partial_2 f - \partial_2 \partial_1 f \end{pmatrix} = 0$$

(5):

$$\nabla\cdot(curl F) = \nabla\cdot\begin{pmatrix}\partial_2 F_3 - \partial_3 F_2\\ \partial_3 F_1 - \partial_1 F_3\\ \partial_1 F_2 - \partial_2 F_1\end{pmatrix} = \partial_1(\partial_2 F_3 - \partial_3 F_2) + \partial_2(\partial_3 F_1 - \partial_1 F_3) + \partial_3(\partial_1 F_2 - \partial_2 F_1) = 0$$

■

2.2 The physical quantities

Let us first introduce the notations for the electric and magnetic fields.

Notation	Unit	Description
$\mathrm{E} = (\mathrm{E}_1, E_2, E_3)^T$	[V/m]	electric field intensity
$\mathrm{D} = (\mathrm{D}_1, D_2, D_3)^T$	[As/m^2]	electric flux density
$\mathrm{H} = (\mathrm{H}_1, H_2, H_3)^T$	[A/m]	magnetic field intensity
$\mathrm{B} = (\mathrm{B}_1, B_2, B_3)^T$	[Vs/m^2]	magnetic flux density
$\mathrm{J} = (\mathrm{J}_1, J_2, J_3)^T$	[A/m^2]	electric current density
$\rho = \rho(x, t)$	[As/m^3]	electric charge density
$\mathrm{M} = (\mathrm{M}_1, M_2, M_3)^T$	[Vs/m^2]	magnetization
$\mathrm{P} = (\mathrm{P}_1, P_2, P_3)^T$	[As/m^2]	electric polarization

E describes the direction and strength of an electric field.
D describes the density of electric field lines through an area A.
H is the same as E but for a magnetic field.
B is the same as D but for magnetic field lines.
J describes the density of a flowing current through a conductor.

2.3 Ampere's Law

An electric current or a changing electric flux through a surface produces a circulating magnetic field around any path that bounds that surface. [2]

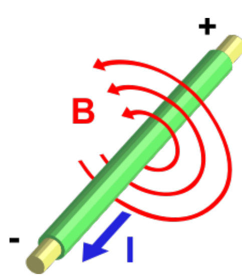

Figure 1: Current running through a wire and creating a magnetic field

Figure 1 illustrates the following integral relation.

$$\oint_{\partial S} H \cdot \tau \, ds = \int_S (J + \frac{\partial D}{\partial t}) \cdot n \, dS$$

Now, using Stokes' Theorem, we have

$$\int_S (curl\, H) \cdot n \, dS = \oint_{\partial S} H \cdot \tau \, ds.$$

Combining the two equations and using Lemma 2.1, we get the first Maxwell-equation

$$curl\, H = J + \frac{\partial D}{\partial t}. \tag{6}$$

2.4 Faraday's Law

Changing magnetic flux through a surface induces an inner voltage in any boundary path of that surface, and a changing magnetic field induces a circulating electric field; see, e.g., [2]. These two physical laws can be described by the following two equations

$$\begin{aligned} u_i &= -\int_S \frac{\partial}{\partial t} B \cdot n \, dS \\ u_i &= \int_S curl\, E \cdot n \, dS \end{aligned}$$

Thus one can easily see how to derive the second Maxwell-equation which is of the following form

$$curl\, E = -\frac{\partial}{\partial t} B. \tag{7}$$

2.5 Electric Gauss' Law

Electric charge produces an electric field, and the flux of that field passing through any closed surface is proportional to the total charge contained within the surface; see, e.g., [2]. Because of this statement we can derive the integral formulation of the law easily. Thus we get that

$$\int_{\partial V} D \cdot n \, dS = \int_V \rho \, dx$$

where ρ is the charge density. Now we use Gauss' Theorem to get

$$\int_V \nabla \cdot D \, dx = \int_V \rho \, dx$$

Since V is an arbitrary area, Lemma 2.1 yields the third Maxwell-equation

$$\nabla \cdot D = \rho \tag{8}$$

2.6 Magnetic Gauss' Law

The total magnetic flux passing through any closed surface is zero; see, e.g., [2]. This is true as long as we have a magnetic north- and southpole. With this statement, we can see that

$$\int_{\partial V} B \cdot n \, dS = 0$$

holds. If one follows with the same steps as for the Electric Gauss' Law, we derive the fourth Maxwell-equation

$$\nabla \cdot B = 0. \tag{9}$$

2.7 Maxwell's Equations

In the previous subsections we derived the following system of partial differential equations:

$$\begin{aligned} curl\, H &= J + \frac{\partial D}{\partial t}, \\ curl\, E &= -\frac{\partial}{\partial t} B, \\ \nabla \cdot D &= \rho, \\ \nabla \cdot B &= 0. \end{aligned}$$

There are also some material laws to get even more relations between each of the physical quantities.

$$B = \mu H + \mu_0 M, \tag{10}$$

$$D = \varepsilon E + P, \tag{11}$$

$$J = J_c + J_i = \sigma(E + v \times B) + J_i; \tag{12}$$

see,e.g., [2]

2.8 Vector-potential Formulation

Now that the equations have been derived and relations between the quantities have been stated we can derive some different formulations in vector form. It is comparable with the notes in [3].
We know that if $\nabla B = 0$ then there exists a vector potential $A \in (C^1(\Omega))^3$ for which it follows that $B = curl A$. Substituting this relation into Faradays' Law we get the following equation

$$curl(E + \frac{\partial A}{\partial t}) = 0$$

Therefore, there exists a scalar potential φ such that

$$E + \frac{\partial A}{\partial t} = -\nabla \varphi$$

holds. Furthermore it can be shown that two more relations hold for some scalar potentials $\varphi, \phi \in C^2(\Omega)$ which are

$$\begin{aligned} B &= curl A = curl \hat{A} \qquad , \hat{A} = A + \nabla \varphi \\ E &= -\frac{\partial A}{\partial t} - \nabla \phi = -\frac{\partial \hat{A}}{\partial t} - \nabla \hat{\phi} \qquad , \hat{\phi} = \phi - \frac{\partial \varphi}{\partial t} \end{aligned}$$

Let us assume one can write $\hat{A}$ as in the equation above, then we can insert this just in the definition and get $curl(A + \nabla \varphi) = curl A + curl \nabla \varphi$ but we know that the second part is equal to zero because of the Lemma in section 1.2. The second equation follows because it holds that ∂ and ∇ are able to switch the order of what is applied first to a scalar potential.
To get to the vector potential formulation we have to make some assumptions about our magnetic and electric field. There is no velocity inside the field $v = 0$, we have also no polarization $P = 0$ and magnetization $M = 0$. With Amperes' Law and express that we can write B as $curl A$, thus getting

$$curl \mu curl \hat{A} + \sigma \frac{\partial \hat{A}}{\partial t} + \varepsilon \frac{\partial^2 \hat{A}}{\partial t^2} = J_i \qquad (13)$$

2.9 Electric-field Formulation

Using Faradays' Law we can see that

$$curl E = -\frac{\partial B}{\partial t} = -\mu \frac{\partial H}{\partial t} - \mu_0 \frac{\partial M}{\partial t}$$

and taking $curl \frac{1}{\mu}$ on both sides of the equation we obtain

$$curl \frac{1}{\mu} curl E = -\frac{\partial}{\partial t} curl H - \frac{\mu_0}{\mu} \frac{\partial}{\partial t} curl M$$

If we now include Amperes', Ohms' Law and simplify it by saying that we have no magnetization and polarization then the electric field formulation follows as

$$curl \frac{1}{\mu} curl E + \sigma \frac{\partial E}{\partial t} + \varepsilon \frac{\partial^2 E}{\partial t^2} = -\frac{\partial J_i}{\partial t} \qquad (14)$$

2.10 Timeharmonic case

One can also use the timeharmonic ansatz to have their partial differential equation not be dependend on time derivatives. For example take U where U can be any of the physical quantities. Take the ansatz

$$U(x,t) = \hat{U}(x) e^{i\omega t} = \hat{U}(x)(cos(\omega t) + i sin(\omega t))$$

If we take the derivative with respect to time we can see that the amplitude $\hat{U}(x)$ is just a constant. It follows that

$$\begin{aligned}
\frac{\partial U}{\partial t} &= (i\omega)\hat{U}(x)e^{i\omega t} = (i\omega)U(x,t) \\
\frac{\partial^2 U}{\partial t^2} &= -\omega^2\hat{U}(x)e^{i\omega t} = -\omega^2 U(x,t).
\end{aligned}$$

We can use this ansatz to get another form of the vector potential formulation

$$curl\, \mu curl\, A + i\sigma - \varepsilon\omega^2 A = J_i$$

but if one wanted to use this ansatz for the electric field formulation, we would have to deal with an imaginary source J_i, because this formulation is nothing else than the derivative of the vector potential formulation with respect to time.

$$curl \frac{1}{\mu} curl E + i\sigma\omega E - \varepsilon\omega^2 E = -i\omega J_i$$

2.11 The Helmholtz equation in vector form for the electric-field formulation

Again we start with Faradays' Law, but this time we have more specifications which are the following: $\mu, \sigma, \varepsilon_0$ shall be constants and we further have no magnetisation.

$$curl E = -\frac{\partial B}{\partial t} = -\mu\frac{\partial H}{\partial t}$$

Now take the same step as in section 2.9 and get the same result which is

$$curl \frac{1}{\mu} curl E + \sigma\frac{\partial E}{\partial t} + \varepsilon\frac{\partial^2 E}{\partial t^2} = -\frac{\partial J_i}{\partial t}$$

Because μ is a constant we can take it out of the *curl* and multiply the whole equation with it to get

$$curl curl E + \sigma\mu\frac{\partial E}{\partial t} + \varepsilon\mu\frac{\partial^2 E}{\partial t^2} = -\mu\frac{\partial J_i}{\partial t}$$

With the Lemma in section 1.2 we can rewrite the *curl curl* in the following form

$$\nabla(\nabla\cdot E) - \Delta E + \sigma\mu\frac{\partial E}{\partial t} + \varepsilon\mu\frac{\partial^2 E}{\partial t^2} = -\mu\frac{\partial J_i}{\partial t}$$

Furthermore we also know that

$$0 = \rho = \nabla\cdot D = \varepsilon_0\nabla\cdot E$$

this relation brings us nearly to the helmholtz equation in vector form

$$-\Delta E + \sigma\mu\frac{\partial E}{\partial t} + \varepsilon\mu\frac{\partial^2 E}{\partial t^2} = -\mu\frac{\partial J_i}{\partial t}$$

the last step is to use the previously discussed timeharmonic ansatz to get

$$-\Delta E + i\sigma\omega E - \varepsilon\omega^2 E = -i\omega J_i \tag{15}$$

2.12 The Helmholtz equation in scalar form

We start by making some fair assumptions which help us to simplify the whole system. The first one is that we have a circular or quadratic base of a domain and the extension in the third direction is much bigger than the directions in either of the other two directions

$$\hat{\Omega} = \Omega \times (-h, h) \in \mathbb{R}^3$$

Our source is an orthonormal vector on our base domain

$$J_i = \begin{bmatrix} 0 \\ 0 \\ J_3(x_1, x_2) \end{bmatrix}, x = (x_1, x_2) \in \Omega$$

Furthermore we can say that $\nabla J = 0$. We only have magnetic and electric fields parallel to our base domain.

$$H = \begin{bmatrix} H_1(x_1, x_2) \\ H_2(x_1, x_2) \\ 0 \end{bmatrix}, x = (x_1, x_2) \in \Omega$$

$$B = \begin{bmatrix} B_1(x_1, x_2) \\ B_2(x_1, x_2) \\ 0 \end{bmatrix}, x = (x_1, x_2) \in \Omega$$

As one can see, the B_3 is equal to zero which means that the following equation is fulfilled.

$$0 = B_3 = (curl\ A)_3 = \partial_2 A_1 - \partial_1 A_2 = 0$$

The next step is to find a matrix A which also holds for that equation and it is easy to see that

$$A = A(x_1, x_2) = \begin{bmatrix} \lambda \\ \lambda \\ A_3(x_1, x_2) \end{bmatrix}, x = (x_1, x_2)$$

fulfills this, moreover λ is only a constant so we can set it to zero to get an even easier matrix to work with.

$$A = A(x_1, x_2) = \begin{bmatrix} 0 \\ 0 \\ A_3(x_1, x_2) \end{bmatrix}, x = (x_1, x_2)$$

Before we continue, there is one aspect one still has to include the Coulomb gauge which says that

$$div(A) = \nabla \cdot A = 0$$

If we know take a look at B as $curl A$ we see that

$$B = curl\ A = \begin{bmatrix} \partial_2 A_3 \\ -\partial_1 A_3 \\ 0 \end{bmatrix}$$

Now to conclude everything we take $curl\, curl A$ and get

$$curl\ (curl\ A) = \begin{bmatrix} 0 \\ 0 \\ -\partial_1(\partial_1 A_3) - \partial_2(\partial_2 A_3) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\Delta A_3 \end{bmatrix}$$

Thus we get our scalar helmholtz equation in the following form

$$-\Delta A_3(x) - (\varepsilon\omega^2 - \sigma(i\omega))A_3(x) = J_3(x) \tag{16}$$

and if we further set $k^2 = \varepsilon\omega^2 - \sigma(i\omega)$ we get the form we will use for the numerical solutions

$$-\Delta A_3(x) - k^2 A_3(x) = J_3(x) \tag{17}$$

3 Numerical solution by finite element method

3.1 Introduction

The goal of this project is to compute an approximate solution of the model of wifi signal generated by a router in a flat by means of the finite element method. For this, we will solve Helmholtz' equation. This equation can be derived from the wave equation by Fourier transformation; see Chapter 2.
This Helmholtz equation can be written in the form

$$-\Delta u - k^2 u = g \text{ in } \Omega.$$

with approriate boundary conditions.
In this equation, we have different data that are given :

- k which is the wave number. Which depends of the permittivity μ_0, the permeability ϵ_0 of the medium, and also of the frequency. Indeed, we have $k = \omega/c$ where $c = 1/\sqrt{\epsilon_0\mu_0}$ is the speed of light.
- g which is the source term.

3.2 Dirichlet boundary value problem

3.2.1 Showing solvability for sufficiently small k

Consider the Dirichlet boundary value problem

$$\begin{cases} -\Delta u - k^2 u &= g \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{cases} \tag{18}$$

for a given source function $g \in L^2(\Omega)$. Find a condition on k which guarantees that (18) has a unique weak solution.

Let v be a sufficiently smooth test function that vanishes on $\partial\Omega$. We multiply (18) by v, and we integrate over Ω, obtaining:

$$-\int_\Omega \Delta uv\,dx - \int_\Omega k^2 uv\,dx = \int_\Omega gv\,dx.$$

Now, we can integrate by parts getting:

$$-\int_{\partial\Omega} \frac{\partial u}{\partial n} v\,ds + \int_\Omega \nabla u \cdot \nabla v\,dx - \int_\Omega k^2 uv\,dx = \int_\Omega gv\,dx \tag{19}$$

and since $v = 0$ on $\partial\Omega$, we have

$$\int_\Omega \nabla u \cdot \nabla v\,dx - \int_\Omega k^2 uv\,dx = \int_\Omega gv\,dx \tag{20}$$

The functions u and v need to have a first-order weak derivative in $L_2(\Omega)$, and the boundary conditions are such that the traces of u and v are 0 on $\partial\Omega$. Thus, V has to be $H_0^1(\Omega)$, equipped with the norm $||v||_V = (||v||^2_{L_2(\Omega)} + |v|^2_{H^1(\Omega)})^{\frac{1}{2}}$. Therefore, the variational problem (20) can be reformed as follows: Find $u \in V = H_0^1(\Omega)$ such that

$$a(u,v) = F(v) \quad \forall v \in V \tag{21}$$

where the bilinear form $a : V \times V \to \mathbb{R}$ and the linear form $F \in V^\star$ are defined as in the following way. $a(u,v) = \int_\Omega \nabla u \nabla v\,dx - \int_\Omega k^2 uv\,dx$ and $F(v) = -\int_\Omega gv\,dx$.

Using the Lax-Milgramm theorem, we show that, for a sufficiently small k, there exists a unique $u \in V$ such that the identity (21) is valid.

Indeed, the form $a(.,.)$ is bilinear. Let $u_1, u_2, v \in H_0^1(\Omega)$ and $\lambda, \mu \in \mathbb{R}$. Then, we have

$$\begin{aligned} a(\mu u_1 + \lambda u_2, v) &= \int_\Omega \nabla(\mu u_1 + \lambda u_2) \cdot \nabla v\,dx - \int_\Omega k^2(\mu u_1 + \lambda u_2) v\,dx \\ &= \mu(\int_\Omega \nabla u_1 \cdot \nabla v\,dx - \int_\Omega k^2 u_1 v)\,dx + \lambda(\int_\Omega \nabla u_2 \cdot \nabla v\,dx - \int_\Omega k^2 u_2 v)\,dx \\ &= \mu a(u_1, v) + \lambda a(u_2, v). \end{aligned}$$

Moreover, the form is symmetric. Indeed,

$$\begin{aligned} a(u,v) &= \int_\Omega \nabla u \cdot \nabla v \, dx - \int_\Omega k^2 uv \, dx \\ &= \int_\Omega \nabla v \cdot \nabla u \, dx - \int_\Omega k^2 vu \, dx \\ &= a(v,u), \end{aligned}$$

and therefore the form $a(.,.)$ is a symmetric bilinear form.

In addition to this, we now show that the bilinear form $a : V \times V \to \mathbb{R}$ is continuous and coercive on V.
Continuity : $\exists M > 0, |a(u,v)| \leq M||u||_V||v||_V, \forall u, v \in V.$
Using triangle inequality and Cauchy-Schwarz

$$(u,v)_{L^2} \leq ||u||_{L^2}||v||_{L^2},$$

we get

$$\begin{aligned} |a(u,v)| &= |\int_\Omega \nabla u \cdot \nabla v \, dx - \int_\Omega k^2 uv \, dx| \\ &\leq |\int_\Omega \nabla u \cdot \nabla v \, dx| + |\int_\Omega k^2 uv \, dx| \\ &\leq ||\nabla u||^2_{L^2}||\nabla v||^2_{L^2} + k^2||u||^2_{L^2}||v||^2_{L^2} \\ &\leq ||u||_V||v||_V + k^2||u||_V||v||_V \\ &\leq C_1||u||_V||v||_V, \end{aligned}$$

with $C_1 = 1 + k^2$.

Coercivity : $\exists \alpha > 0$ such that $|a(u,u)| \geq \alpha||u||^2_V, \forall u \in V.$
Using Friedrichs' inequality

$$||u||_{L_2(\Omega)} \leq c_F|u|_{H^1(\Omega)} \, \forall u \in V = H^1_0(\Omega),$$
$$\frac{1}{c_F^2} \leq inf_{v \in H^1_0(\Omega)} \frac{\int_\Omega \nabla u \cdot \nabla u \, dx}{\int_\Omega u^2 \, dx} = \lambda_{min}(-\Delta)$$

we obtain

$$
\begin{aligned}
a(u,u) &= \int_\Omega |\nabla u|^2\,dx - \int_\Omega k^2u^2\,dx \\
&= |u|^2_{H^1} - k^2 \frac{1}{\frac{\int_\Omega |\nabla u|^2\,dx}{||u||^2_{L^2}}} |u|^2_{H^1} \\
&\geq |u|^2_{H^1} - k^2 \frac{1}{inf_{v\in H^1_0(\Omega)} \frac{\int_\Omega \nabla v \nabla v\,dx}{\int_\Omega vv\,dx}} |u|^2_{H^1} \\
&= (1 - \frac{k^2}{\lambda_{min}(-\Delta)})(\frac{1}{2}|u|^2_{H^1} + \frac{1}{2}|u|^2_{H^1}) \\
&\geq (1 - \frac{k^2}{\lambda_{min}(-\Delta)})(\frac{1}{2}|u|^2_{H^1} + \frac{1}{2c_F^2}||u||^2_{L^2}) \\
&\geq (1 - \frac{k^2}{\lambda_{min}(-\Delta)}) min\{\frac{1}{2}, \frac{1}{2c_F^2}\}||u||^2_{H^1} \qquad (22)
\end{aligned}
$$

where the factor $(1-(k^2)/(\lambda_{min}(-\Delta)))$ is larger than zero for all $k^2 < \lambda_{min}(-\Delta)$.

According to these three points, by the Lax-Milgram theorem, there exists a unique $u \in V$ such that $a(u,v) = F(v), \forall v \in V$, provided $k^2 < \lambda_{min}(-\Delta)$. Now the question arises what happens if $k^2 \geq \lambda_{min}(-\Delta)$. If $k^2 \geq \lambda_{min}(-\Delta)$, then we can not use Lax-Milgram, because the requirement for coercivity is not fulfilled.

3.2.2 Further Analysis for a sample example

Lax-Milgram only holds up until the first eigenvalue, $\lambda_{min}(-\Delta)$ of the system $-\Delta u = \lambda u$ with $u = 0$ on $\partial\Omega$. First we compute the eigenvalues in a special example. Therefore we solve our aforementioned equation on a rectangular (0,1) $\times$ (0,1) domain. Taking the ansatz for separation of variable $u(x,y) = X(x)Y(y)$, we get

$$
\begin{aligned}
-X''(x)Y(y) - Y''(y)X(x) &= \lambda X(x)Y(y) \\
\implies -\frac{X''(x)}{X(x)} - \frac{Y''(y)}{Y(y)} &= \lambda \\
\implies -\frac{X''(x)}{X(x)} &= \lambda + \frac{Y''(y)}{Y(y)}
\end{aligned}
$$

Now the LHS is independent of y and the RHS is independent of x. Therefore it follows that there exists a constant c for which

$$
-\frac{X''(x)}{X(x)} = \lambda + \frac{Y''(y)}{Y(y)} = c
$$

. Now we can solve the two equations independently

$$-X''(x) = cX(x)$$
$$-Y''(y) = (c - \lambda)Y(y)$$

Solving the first equation and inserting $X(0) = X(1) = 0$, yields that $c = \pi^2 n^2$ for $n \in \mathbb{N}$. If we solve the second equation in the same way with similar boundary values we finally come to the conclusion that $\lambda_{nm} = \pi^2(n^2 + m^2)$ for $n, m \in \mathbb{N}$. Thus our exact solution is of the form $u(x, y) = sin(n\pi x)sin(m\pi y)$. The following pictures are a comparison on what happens for the approximate solution in the program with k^2 being away and close to the first eigenvalue.

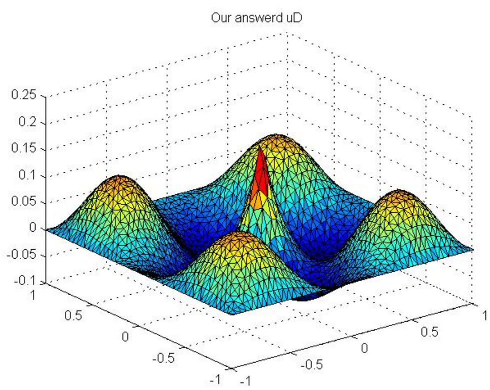

Figure 2: $k^2 \ll \lambda_{min}$

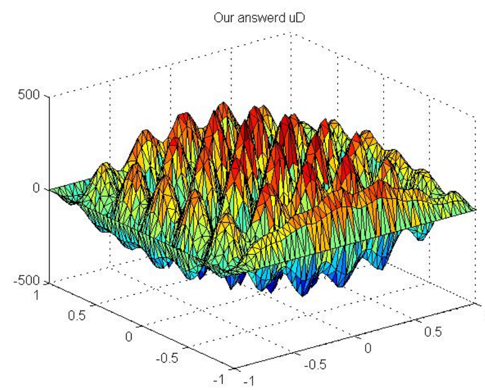

Figure 3: $k^2 \approx \lambda_{min}$

In order to discuss the general case, we need the Fredholm theory; see,e.g., [4] , which will not be discussed in this work. The solvability of the corresponding finite element equations is discussed later in subsection 3.3.1

3.3 Finite Element Discretization

The function u in the Helmholtz equation can be approximated by a function u_h using linear combinations of basis functions (ψ_i) in the following form.

$$u \approx u_h$$
$$u_h = \sum_i u_i \psi_i$$

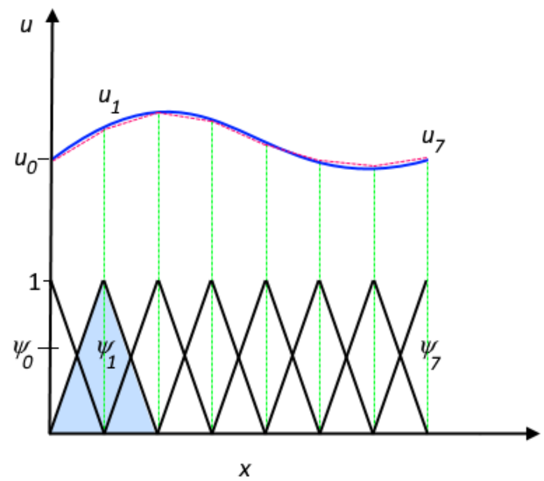

Figure 4: Using combinations of finite element basis functions to approximate the function u

Here the solid blue line is approximated by the red dotted line which in this case is our u_h. A benefit of using FEM is, it does not have to be uniformly grid as we can see in the following figure. We can discretize the domain in relation to the gradient of our function. If we have at some point a steep drop or increase in the function, we can adapt increments to decrease the likelihood of a large difference between the function u and its approximation u_h.

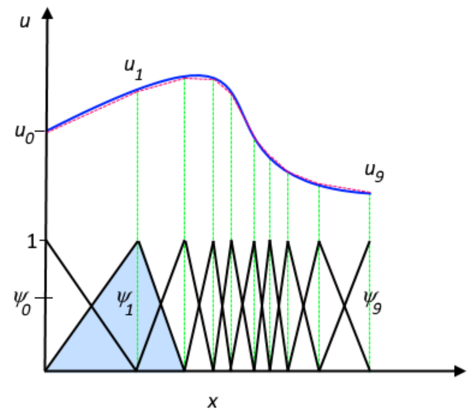

Figure 5: Using combinations of finite element basis functions to approximate the function u

In 2D, we first have to discuss what to use now, instead of the intervals on the x-axis we used in the 1D case. The approach we use in this project is to put our domain together via triangles.
Of course, those triangles have to fulfill certain properties to be considered for the triangulation. The first property is a topological one. The triangulation has to be conforming, by this we mean that the intersection of two triangles is either a node or the common edge of both. The second property is a geometric one, meaning $h_k/\rho_k \leq \beta$ with h_k being the longest edge of the triangle, ρ_k being the diameter of the inscribed circle and $\beta \geq 1$.

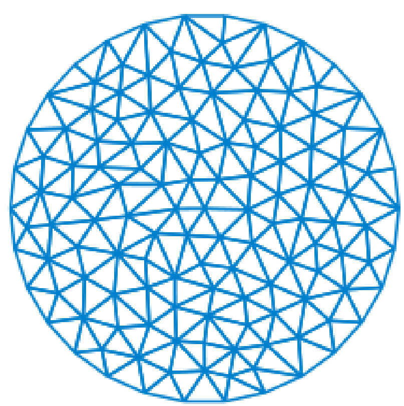

Figure 6: Conforming 2D mesh for a circle

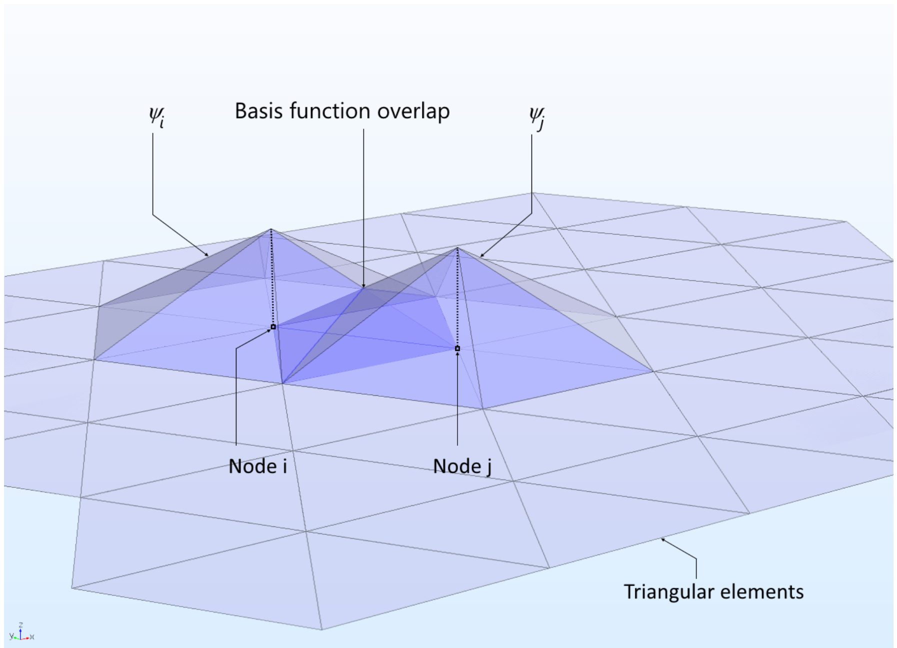

Figure 7: Showcasing two different witches hat functions

The figure above shows the basis functions, which we have used in our project. We called this ones "witches hat functions" because they have nearly the same appearence Now we can make a similar approach as in the 1D case where we approximated $u \approx u_h = \sum_k u_i \psi_i$.
Our weak formulation was of the following form

$$\int_\Omega \nabla u \cdot \nabla v \, dx - \int_\Omega k^2 uv \, dx = \int_\Omega gv \, dx. \tag{23}$$

So now instead of u and v we are going to use their approximations which leads to the following

$$\sum_{\Omega_k \in \mathcal{T}} (\sum_{i,j} u_i v_j \int_{\Omega_k} \nabla\psi_i \cdot \nabla\psi_j - \sum_{i,j} u_i v_j \int_{\Omega_k} k^2 \psi_i \psi_j) = \sum_{\mathcal{T}_k \in \Omega} \sum_j v_j \int_{\Omega_k} g\psi_j.$$

where $\mathcal{T}_k$ is the set of all triangles in the domain Ω.
The next step is to transform this equation into a matrix equation which then can be solved by a computer program. We used MATLAB in our project. Splitting the left hand side up into two parts, the first part being the so called stiffness matrix $A_h = (A_{ij})$

$$A_{ij} = \int_\Omega \nabla\psi_i \nabla\psi_j \, dx$$

and the second part being the mass matrix $M_h = (M_{ij})$

$$M_{ij} = \int_\Omega \psi_i \psi_j \, dx$$

We note that the matrices A_h, M_h are sparse everywhere, where the two basis functions ψ_i, ψ_j do not overlap.
For the right hand side, it is even easier. Because of the form of our source, which is a dirac function multiplied with a basis function on one triangle, it is everywhere sparse except where the basis function ψ_j coincides with ψ_s. Where ψ_s is the basis function on the triangle where our source is.

3.3.1 The finite element equations

$$(A_h - \lambda M_h)u_h = g_h \tag{24}$$

We can show existence and uniqueness by showing $det(A_h - \lambda M_h) \neq 0$. Since A_h and M_h are SPD matrices. Rewrite M_h as $P^{-1}DP$ and wrtie $D = D^{0.5} * D^{0.5}$ thus we can rewrite the discrete equation in the following form, where $M_h^{-0.5} = (PD^{0.5})^{-1}$. Multiply (24) by $M_h^{-0.5}$, we get

$$(M_h^{-0.5} A_h M_h^{-0.5} - \lambda I)(M_h^{0.5} u_h) = M_h^{-0.5} g_h$$

Now having $\hat{A_h}$,which is again a SPD matrix, instead of $M_h^{-0.5} A_h M_h^{-0.5}$ and $\hat{u_h}$, $\hat{g_h}$ in a similar fashion. Existence and uniqueness can be shown iff

$$det(\hat{A_h} - \lambda I) \neq 0$$
$$\hat{A_h}\hat{u_h} = \lambda \hat{u_h} \, , \lambda \neq EV$$

Now if $\lambda = \lambda_i$ (single EV) then $(\hat{A_h} - \lambda_i I)\hat{u_h} = \hat{g_h}$ is solvable iff $(\hat{g_h}, e_i) = 0$ Meaning $\hat{u_h} = u_h^* + \mu e_i$, if we have multiple EVs $\hat{u_h} = u_h^* + \sum_{i \in EV} \mu_i e_i$, leads to

$$(\hat{A_h} - \lambda_i I)u_h^* = \hat{g_h}$$
$$(\hat{A_h} - \lambda_i I)(\mu e_i) = 0$$

LHS $\implies$ RHS (using symmetry of the matrix)

$$(\hat{A_h} - \lambda_i I)\hat{u_h} = \hat{g_h}$$
$$(\hat{g_h}, e_i) = ((\hat{A_h} - \lambda_i I)\hat{u_h}, e_i) = (\hat{u_h}, (\hat{A_h} - \lambda_i I)e_i) = 0$$

RHS $\implies$ LHS

$$\hat{g_h} = \sum_j \beta_j e_j = \sum_{j \neq i} \beta_j e_j \, , \hat{u_h} = \sum_j \alpha_j e_j$$
$$(\hat{A_h} - \lambda_i I)\hat{u_h} = \sum_j \alpha_j (\hat{A_h} - \lambda_i I)e_j = \sum_j \alpha_j (\lambda_j - \lambda_i)e_j = \sum_j \beta_j e_j$$
$$\iff \alpha_j = \frac{\beta_j}{\lambda_j - \lambda_i} \, , \hat{u_h} = \sum_{j \neq i} \frac{\beta_j}{\lambda_j - \lambda_i} e_j + \mu e_i$$

Thus the system is solvable.

3.3.2 Some further properties

- The finer the mesh, the better our final results. Meaning if we have an area in our 2D domain where the gradient of our function changes a lot, it would be a good idea to refine the mesh in this area as well, similar to the figure for the 1D case
- We know, for k^2 not being an eigenvalue, the matrix $(A - k^2 M)$ has full rank. Therefore, the inverse exists and the system $(A - k^2 M)U = b$ has a solution.
- For a lot of unknown variables the direct approach, just inverting the matrix, which we used in this project, is not really feasible. Therefore, people use iterative methods to solve the system.
- If the PDE is non-linear then the resulting system is also non-linear, this system is for example solvable with the Newton method.

3.4 Code

3.4.1 Real case

In the first step, we try to code the solution with $k \in \mathbb{R}$.

- We generate our flat with $GetPlate$ which returns a square with a mesh. On each side of the square, we assume that there is Nq nodes, so the total of nodes in the square is Nq^2.
- We generate the source as a Dirac function on the coordinates that we choose: we decide on the closest, to the source term, node . The source term appears on the right hand side (our F).
- We use the Dirichlet conditions for the boundaries, with $u(x) = 0, \forall x \in \partial\Omega$.

3.4.2 Scheme of the code

- **Main**

  Choose where to place the walls.

  Extend this matrix to obtain our all flat. Obtain Nq from this matrix.

  Place the source.

  Choose the physical values. For instance the frequency, the permeability and the permittivity.

- **Call stiffness function**
  - Call $[p, tri1, edge] = GetPlate(Nq)$.
  - Sort the mesh (tri).

- **Calculation of the stiffness matrix**
    * Find the closest node from the source coordinates.
    * Loop over all triangles.
    * Loop over each node (2 times if you are in the complex case - cf lower).
    * Calculate the basis functions and their gradients at the nodes.
    * Calculate the evaluation of the solution between the two nodes.
  - Calculate the right hand term (the evaluation of the source term).

- Add the boundary conditions (0 everywhere on the boundary for Dirichlet boundary condition).

- Solve AU=b where U is the solution.

- Plot U.

3.4.3 How to create and to move the term source

To obtain a wifi signal, a router is needed. We represent this router by a Dirac peak placed on a node of the mesh.

- We choose the coordinates for the source term. We find the closest node to the source term and we let this node be the emplacement of the source term.

- We let the right hand term be the source term. Since now, it contains only the values on the triangles around the source term. We let all the other values be zero.

- The source function is represented as a smeared Dirac function placed on the node of the source.

- In the stiffness function, if the triangle has one of its nodes commune with a node of the source term we integrate the dot product between the Dirac function and the basis function.

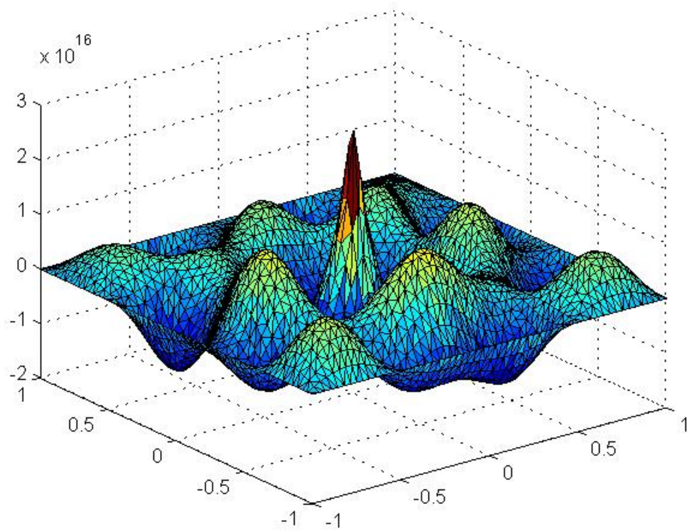

Figure 8: Source at (0,0).

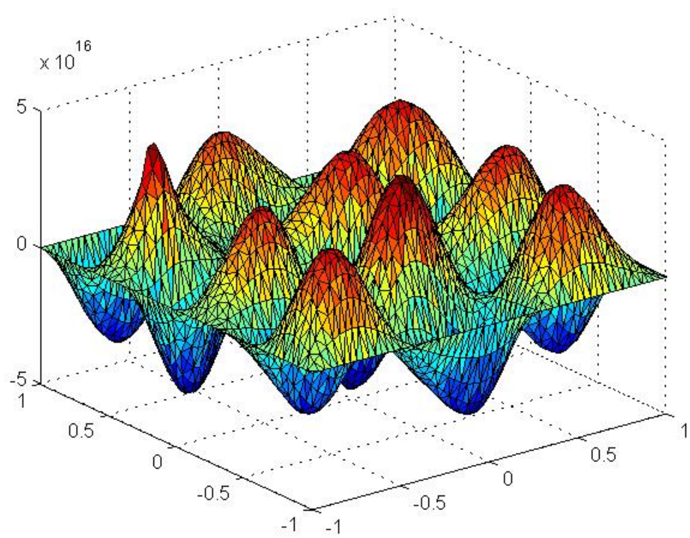

Figure 9: Source at (-0.7,0.5).

3.4.4 How to built walls

Here, the goal is to add walls in our flat. Physically, a wall appears on the permittivity and on the permeability terms. But we can approximate these changes on the coefficients just multiplying k by a unique coefficient $coef\_k$.

Our approach in the code is the following idea :

- We start with a small matrix Mat_built which is the representation of our flat. We put 1 when the medium is "vacuum", 10^4 for example when is concrete.
- We extend this matrix to obtain a matrix which has a sufficient size $Nq*Nq$ to built a mesh of the size $Nq*Nq$ too.
- We build our mesh and we sort the triangles as tri.
- We build a vector $coef_k$ to contain the coefficients of the different media which has the same size than tri. We fill this vector with the values of expanded Mat_built. Two triangles on the same square will have the same value.

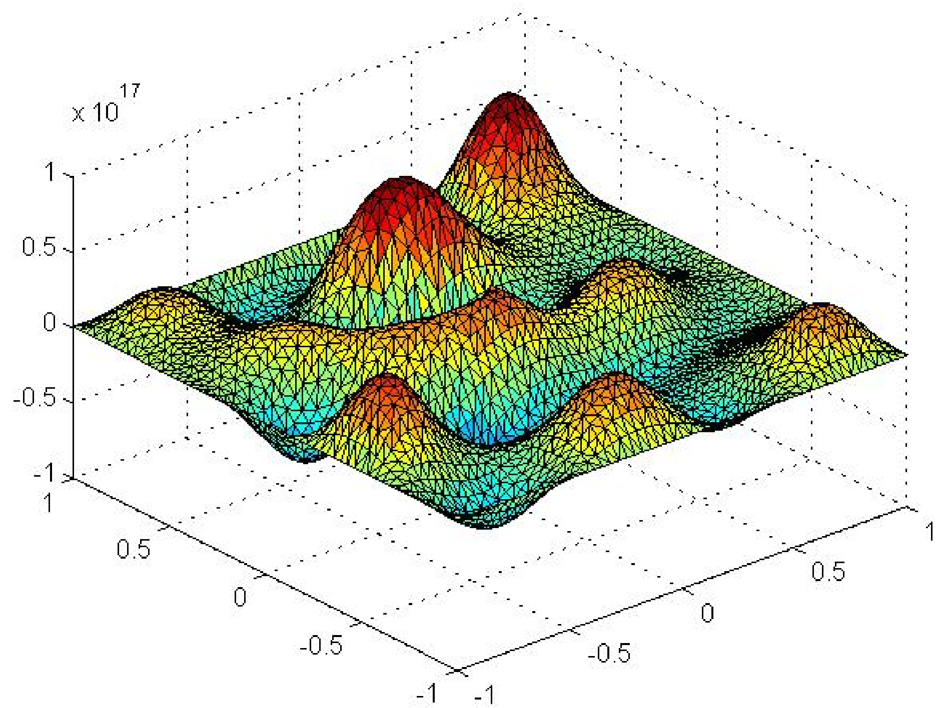

Figure 10: Walls in the 2nd and 6th row.

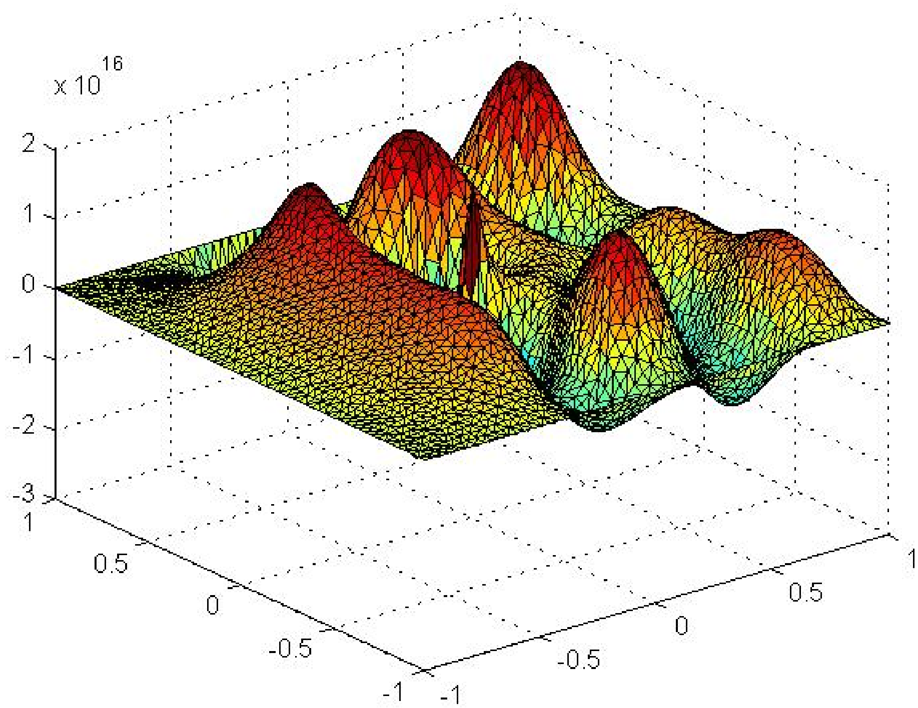

Figure 11: Walls on the first two columns.

We can see that the wall squeezes the oscillations. We can play with the value of the coefficient $coef\_k$ to reduce or increase this phenomeon.

3.4.5 Complex case

Now, we take $k \in \mathbb{C}$. So k has two parts now: a real one and an imaginary one. From now on u is also a complex function.
Let $u = u_1 + iu_2$ and $k = k_1 + ik_2$.

$$\begin{cases} -\Delta(u_1 + iu_2) - k^2(u_1 + iu_2) & = & g \text{ in } \Omega \\ u & = & 0 \text{ on } \partial\Omega \end{cases} \tag{25}$$

If we look at this equation :

$$-\Delta u - k_1^2 u + k_2^2 u - 2ik_1k_2u = g$$

which is equivalent to :

$$-\Delta u_1 - i\Delta u_2 - k_1^2 u_1 - ik_1^2 u_2 + k_2^2 u_1 + ik_2^2 u_2 - 2ik_1k_2u_1 + 2k_1k_2u_2 = g$$

we can separate the real and the imaginary part to obtain two different equations which are linked :

$$\begin{cases} -\Delta u_1 - (k_1^2 - k_2^2)u_1 + 2k_1k_2u_2 &= g \\ -\Delta u_2 - (k_1^2 - k_2^2)u_2 - 2k_1k_2u_1 &= 0 \end{cases}$$

Hence, our goal will be to solve both equations at the same time.
If we integrate by parts we obtain (with the notation of the code) for the real equation :

$$\begin{aligned} \int \nabla u_1 \psi_j \, dx - \int (k_1^2 - k_2^2) u_1 \psi_j \, dx + \int 2k_1k_2 u_2 \psi_j \, dx &= \\ \Leftrightarrow \sum_{n=1}^{3} a_n \int \nabla \psi_n \cdot \nabla \psi_j \, dx - \sum_{n=1}^{3} a_n \int (k_1^2 + k_2^2) \psi_n \psi_j \, dx + \sum_{n=4}^{6} a_n \int 2k_1k_2 \psi_n \psi_j \, dx &= 0 \end{aligned}$$

and for the complex equation :

$$\begin{aligned} \int \nabla u_2 \psi_j \, dx - \int (k_1^2 - k_2^2) u_2 \psi_j \, dx - \int 2k_1k_2 u_2 \psi_j \, dx &= 0 \\ \Leftrightarrow \sum_{h=1}^{3} a_h \int \nabla \psi_n \cdot \nabla \psi_j \, dx - \sum_{n=1}^{3} a_n \int (k_1^2 + k_2^2) \psi_n \psi_j \, dx - \sum_{n=4}^{6} a_n \int 2k_1k_2 \psi_n \psi_j \, dx &= 0 \end{aligned}$$

We need two times more of ψ_j and ψ_h basis functions. In the code, we differentiate the case where ψ_j (resp. ψ_h) is complexe checking the value of the index : if it is strictly bigger than 3; we know that we are on the complex case. If not, we are on the real case.

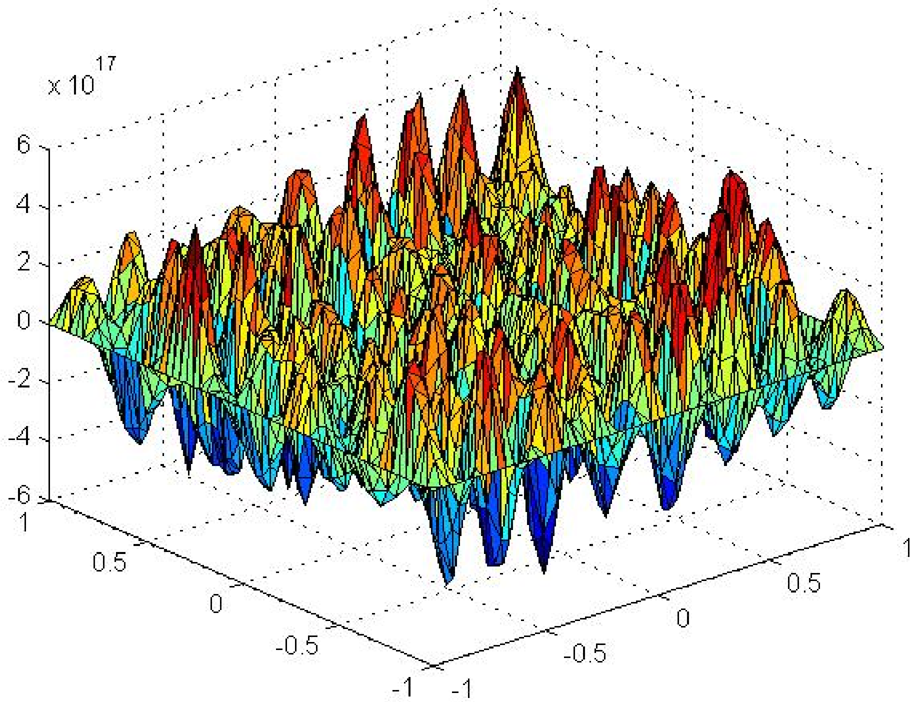

Figure 12: Complex case - Source term at (0,0) - no walls.

We can also add walls in the complex case :

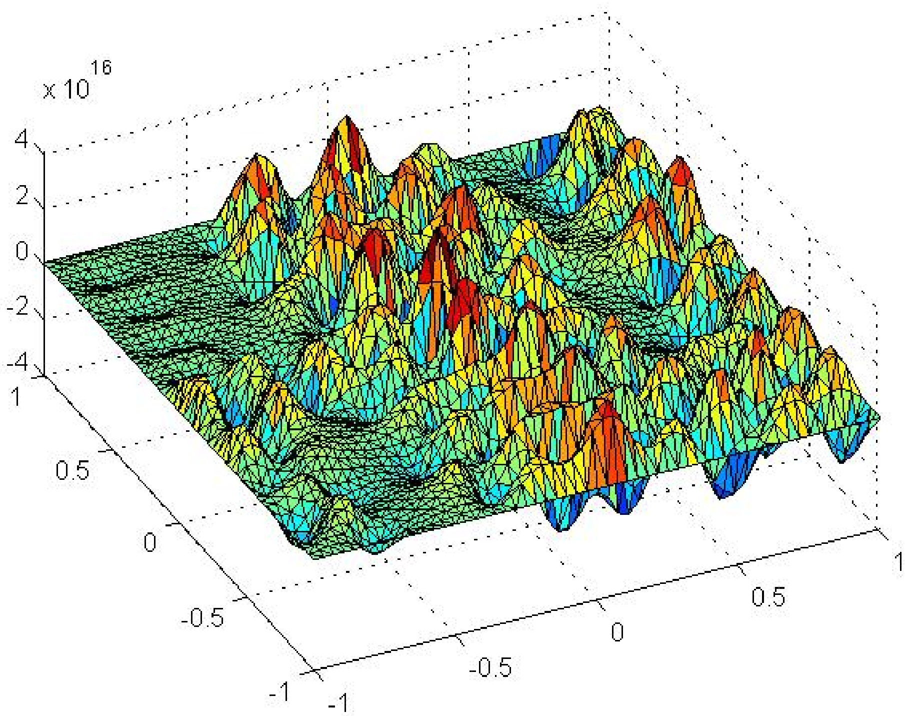

Figure 13: Complex case - Source term at (0,0) - walls.

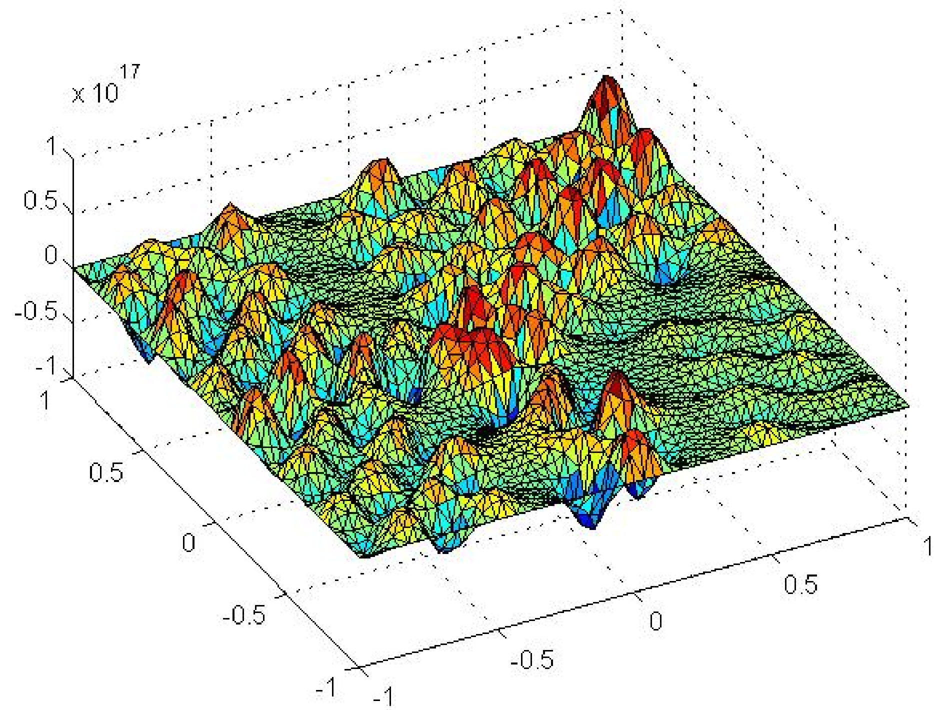

Figure 14: Complex case - Source term at (0,0) - walls.

There is no wall on the bottom right, but the signal still decays because the wave is affected by going through the wall.

3.5 Impedance boundary condition

With the Impedance boundary condition, (18) becomes :

$$\left\{ \begin{array}{rcl} -\Delta u - k^2 u &=& g \text{ in } \Omega \\ \frac{\partial u}{\partial n} - iku &=& 0 \text{ on } \partial\Omega \end{array} \right. \tag{26}$$

We obtain a new weak formulation from (26) :

$$\begin{array}{rcl} -\int_{\partial\Omega} \frac{\partial u}{\partial n} v\, ds + \int_{\Omega} \nabla u \cdot \nabla v\, dx - \int_{\Omega} k^2 uv\, dx &=& \int_{\Omega} gv\, dx. \\ -\int_{\partial\Omega} ikuv\, ds + \int_{\Omega} \nabla u \cdot \nabla v\, dx - \int_{\Omega} k^2 uv\, dx &=& \int_{\Omega} gv\, dx. \end{array}$$

Hence, we arrive at the variational problem.
Find $u \in H^1(\Omega)$ such that

$$-\int_{\partial\Omega} ikuv\,ds + \int_{\Omega} \nabla u \cdot \nabla v\,dx - \int_{\Omega} k^2 uv\,dx \quad = \quad \int_{\Omega} gv\,dx \quad \forall v \in H^1(\Omega).$$

This leads to

$$-ik \sum_{j\in\omega_h} a_j \int_{\partial\Omega} \psi_j\psi_h\,ds + \sum_{j\in\omega_h} a_j \int_{\Omega} \nabla\psi_j \cdot \nabla\psi_h\,dx - k^2 \sum_{j\in\omega_h} a_j \int_{\Omega} \psi_j\psi_h\,dx = \int_{\Omega} g\psi_h\,dx.$$

in the finite element discretization, where ψ_j and ψ_h are the basis functions of the Lagrange space and ω_h being the set of all indices.

Unfortunately, problems on the code with dimensions impeded us to compute the code with the Impedance boundary condition. But the solution for the waves should be similar as in the previous cases, decreasing from the source term to the boundaries. The only additional thing is that waves get reflected back into the domain at the edges of the domain.

3.6 Extras

We have implemented the code in the real case with Dirichlet boundary conditions with a round flat. We can observe something really similar at waves when we throw a rock on water. Moreover, we can see that the amplitude of the waves is decreasing with the distance from the source. However the shape of the waves is not really regular. We can assume that it is due to the Dirac distribution which is not really regular.

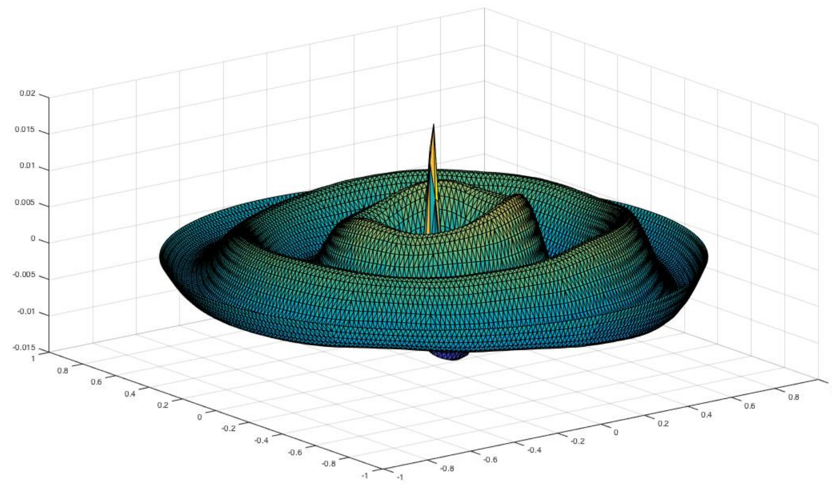

Figure 15: Round flat

4 Summary and outlook

We started by deriving Mawell's equations from some physical properties we experience in our everyday lives. With those and some dependencies between each physical quantities we were able to get to the Helmholtz equation in vector and scalar form. Afterwards a numerical solution, via the finite element approach, was implimented to calculate the amplitude of the solution on every node on our grid. During this time we learned a lot about implementing the FEM-method in Matlab and about how to approach a project with very little to no guidelines whatsoever.

No code is perfect and you can always update it to make it better. What we would do to make our code better if we would have had more time, or what would have been nice if we implemented it is following:

- Actually implement the Impedance boundary condition.
- Adapt our Matrix for the wall in a way that we can assign every triangle another coefficient, this would mean we could adapt our flat even more and have diagonal walls aswell.
- Use the vectorisation of Matlab to obtain a faster code.

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