Adaptive Multilevel Finite Element Analysis of Elastic Body-Body Contact Problems

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Abstract

This thesis is concerned with the modelling and efficient solving of a body-body contact problem without friction. After a short introduction to the fields of continuum mechanics and linear elasticity, the *Signorini-* or *Geometrical-* contact condition is derived. The resulting problem is brought into a variational form. Due to the *Signorini-* or *Geometrical-*contact condition, the variational formulation is a variational inequality. Thus the body-body contact problem isn't linear even if linear elastic problems are considered. For this variational inequality several equivalent formulations are denoted, which enables the usage of different abstract tools. With these tools existence and uniqueness of the solution of the body-body contact problem is proved under reasonable assumptions.

In order to solve the body-body contact problem numerically, a finite element formulation is derived. For this finite element formulation, the *Signorini-* or *Geometrical-*contact condition results in *nodal constraints*. If additional regularity of the solution of the continuous problem is assumed convergence results are proved.

Due to the large scale of some body-body contact problems and due to missing a-posterior error estimates for the body-body contact problem with *nodal constraints* an ad-hoc refinement strategy is presented.

In order to solve the body-body contact problem efficiently, a solver with optimal time complexity is constructed.

In addition to the body-body contact problem a simple *Laplace* problem with a non-matching grid is analysed for *nodal constraints*. The main result for this *Laplace* problem with *nodal constraints* is that it's possible to achieve an optimal convergence result in the primal variable, if a consistent mesh dependent stabilisation term is added to the original formulation.

Zusammenfassung

Dies Arbeit beschäftigt sich mit der Modellierung und dem effizienten Lösen eines Körper-Körper-Kontaktproblems ohne Reibung. Nach einer kurzen Einführung in das Gebiet der Kontinuumsmechanik und der linearen Elastizitätstheorie, wird die *Signorini-* oder auch *Geometrische-* Kontaktbedingung abgeleitet. Das daraus entstehende Problem wird danach in eine Variationsformulierung überführt. Aufgrund der *Signorini-* oder *Geometrischen-*Kontaktbedingung, ist das Variationsproblem eine Variationsungleichung. Daher ist das Körper-Körper-Kontaktproblem nichtlinear, auch wenn ein linear elastisches Problem betrachtet wird. Für diese Variationsungleichung werden verschiedene äquivalente Formulierungen betrachtet, welche die Nutzung abstrakter Werkzeuge erlauben. Mit diesen Hilfsmitteln kann die Existenz und die Eindeutigkeit des Körper-Körper-Kontaktproblems, unter vernünftigen Voraussetzungen an die Glattheit der Lösung, bewiesen werden.

Um das Körper-Körper Kontaktproblem numerisch zu lösen, wird eine Finite Element Formulierung eingeführt. Für die in dieser Arbeit verwendete Finite Element Formulierung werden die *Signorini-* oder auch *Geometrische-* Kontaktbedingungen zu *Knotenrestriktionen*. Für diese Art der Diskretisierung werden Approximationsresultate gezeigt. Leider sind die Approximationsresultate nur gültig, wenn zusätzliche Regularität der Lösung vorausgesetzt wird.

Aufgrund der geometrischen Begebenheiten einiger Körper-Körper-Kontaktproblemen und aufgrund fehlender Resultate über a-posteriori Fehlerschätzer für das Körper-Körper-Kontaktproblem mit *Knotenrestriktionen*, wird ein ad-hoc Verfeinerungsstrategie angegeben.

Abschließend wird ein Verfahren zum effizienten lösen des Körper-Körper- Kontaktproblems konstruiert, das optimal in der Zeit ist.

Zusätzlich zum Körper-Körper-Kontaktproblem wird ein einfaches *Laplace*-Problem für nichtkonforme Netze und *Knotenrestriktionen* analysiert. Die Hauptaussage für dieses *Laplace*-Problem ist, das es möglich ist auch für *Knotenrestriktionen* optimale Konvergenzeigenschaften für die primale Variable zu erhalten, wenn das Originalproblem durch einen konsistenten und netzabhängigen Stabilisierungsterm ergänzt wird.

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Insofern sich die Sätze der Mathematik auf die Wirklichkeit beziehen sind sie nicht sicher, und insofern sie sicher sind beziehen sie sich nicht auf die Wirklichkeit

Albert Einstein

Contents

Chapter 1

Introduction

Boundary value problems involving contact are of great importance in industrial applications in mechanical and civil engineering. The range of application includes metal forming processes, drilling problems, bearings, crash analysis of cars, car tires or cooling of electronic devices.

From the modelling of the body-body contact problem to the numerical simulation, there are a lot of steps between, having it's own difficulties.

Starting with the modelling, the first difficulty consists in describing the non-penetration of the bodies and the correct transfer of forces. This can't be done, excepted of one special case, the case that one body is a rigid plane, exactly. For contact problems undergoing small deformations, the non-penetration condition is approximated by the so called *Signorini-condition*. For large deformations it's still unsolved how to approximate the non-penetration condition. Some try to solve this problem by reducing the large deformations into a number of small ones. This is done by considering the contact problem dynamically or at least quasi static. For modelling the contact problem refer to the standard book for contact mechanics Kikuchi and Oden [34]. Other references concerned with the modelling of the body-body contact problem are Boieri, Gastaldi and Kinderlehrer [7], Eck [22]. In Carstensen, Scherf and Wriggers [15] a very general approximation of the non-penetration condition, in the context of *Signorin-condition*, is presented. The *Signorini-* contact condition usually results into variational inequalities.

After having a model it's necessary to check the solvability of this model. Because the contact problem usually results into variational inequalities, there are powerful tools available. Nevertheless the solvability isn't guaranteed in all cases. Especially contact formulations with non-linear materials or considering contact with friction, the solvability becomes difficult. A major problem is to guarantee that the contact problem is stable, i.e. that the solution exists, is unique and depends continuously on the data. A lot of abstract results to handle variational inequalities are presented in Eckland and Temam [23], Showalter [46], Kinderlehrer and Stampacchia [35], or every other book concerned with the calculus of variation. Nevertheless some contact problems don't fit into these abstract results. Thus there are many references especially concerned with the contact problem. Eck [22] especially concerns contact with friction. In Kikuchi and Oden [34], Haslinger, Hlaváček and Nečas [29] contact problems with non-linear material laws are considered as well as contact with friction. That the down sized elastic body-body contact problem without friction can become ill posed is proved in Carstensen, Scherf and Wriggers [15].

Having a model and the existence of a solution it's necessary to think about the solution itself. Only a few contact problems are solvable analytically. The most famous analytic solution of a body-body contact problem was given by Hertz [30], who considered the elastic contact of two circles. Due to the difficulty of the contact problem it's necessary to solve it numerically. Note that the contact problem is non-linear even when the material is linear elastic. First of all the contact problem has to be discretised. There are at least two possibilities in the contents of FEM. The first method is to discretise the contact condition by *nodal constraints*, i.e. the non-penetration condition and the transfer of forces are only done for nodes at the contact zone and not for the hole contact zone. Indeed it seems so that the *nodal constraints* don't fit into the (mixed) FEM approach, but this isn't the case. The draw back of *nodal constraints* are the missing convergence results for body-body contact problems without any further regularity. For the *Signorini* problem *nodal constraints* are well suited, because this is nothing else than a matching grid. For this case convergence results are proved in Kikuchi and Oden [34], Haslinger, Hlaváček and Nečas [29] Brezzi, Hager and Raviart [13] [14] and Falk [24]. In Wriggers [51] some different discretisations are presented which results into *nodal constraints*. It was mentioned that the *Signorini* problem discretised with *nodal constraints* behave like a matching grid, whereas the body-body contact problem behaves like a non-matching grid, despite of some obvious special cases. Thus the second method, developed in the last decade, for discretising the constraints is the *Mortar* method. This method was considered by Belgacem, Hild and Laborde [6] [5], which proved convergence, but not optimal convergence.

Due to large scales of some contact problems it's useful to use adaptive refinement strategies. For variational equations there are a lot of a-posteriori error estimators which enable adaptive refinement. For variational inequalities the literature concerning a-posteriori error estimators is sparse (e.g. Kornhuber [38], Ainsworth, Oden and Lee [3]). The presented a-posteriori error estimators are usually not applicable to the body-body contact problem, they are only well suited for the *Signorini* problem. Thus some ad-hoc error estimators have to be used. Some of this ad-hoc error estimators are presented by Verfürth [49] [50] or Ainsworth and Oden [2]. In Carstensen, Scherf and Wriggers [15] an a-posteriori error estimator, constructed for the body body contact problem, and a solving algorithm based on penalisation, is presented.

For solving the discretised contact problem a lot of algorithms are available. Due to the non-linearity of the contact problem most of the algorithms are from non-linear programming. An overview for these algorithms is presented in Glowinski [26]. In the last decade a lot of algorithms, for solving contact problems are published, based on iterative methods, like Hackbusch and Mittelmann [27], Hoppe and Kornhuber [32], Hoppe [33], Brandt and Cryer [11], Mandel [40], Kornhuber [36] [37], Tarvainen [47] [48], Dostaăl, Gomes Neto and Santos [20], Schöberl [43].

Overview

This thesis is only concerned with a body-body contact problem without friction. The aim is to develop a time optimal algorithm for solving the body-body contact problem.

- Chapter 2 is concerned with the modelling of the non-penetration condition (*Signorini condition*) as well as the modelling of the transfer of contact forces for the body-body contact problem.

- Chapter 3 is a short introductory into *Sobolev spaces* and there representation as interpolation spaces. Furthermore a partial ordering for *Sobolev spaces* is introduced which will be important to derive a *weak* formulation for the body-body contact problem.

- Chapter 4 is concerned with the derivation of the *weak* formulation of the body-body contact problem as well as to prove existence and uniqueness for this formulation.

- Chapter 5 gives an introduction into the topic of finite elements. Especially *Scott-Zhang* type interpolation operators are introduced, which play an essential role on the following chapters. Also some abstract existence and uniqueness results as well as some convergence results are presented.

- Chapter 6 presents an analysis of *nodal constraints* for a simple *Lapace* problem with non-matching grids. The goal of this chapter is the proof of an optimal convergence result for *nodal constraints* in the primal variable. This result was also verified by an numerical example.

- Chapter 7 deals with analysis of the discrete body-body contact problem. The discretisation of the constraints was done by *nodal interpolation*. For this discretisation convergence results are proved. The draw back will be, that all these convergence results need additional regularity for the solution. To allow adaptive refinement an ad-hoc mesh refinement strategy is presented.

- Chapter 8 is concerned with the construction of a time optimal algorithm to solve the body-body contact problem.

- Chapter 9 presents numerical results for both, academic examples and a real life problem, the sag of a roll stack (3D)

Chapter 2

Modelling of the Body-Body Contact Problem

In many practical situations in solid mechanics it's important to model the situation of two or more bodies coming into contact with each other. The aim of this chapter is to derive the contact condition for two elastic bodies undergoing small deformations. For the sake of simplicity, this thesis is restricted to contact without friction.

An introduction into the mechanics of continua is presented in CIARLET [17]. The modelling of the body-body contact problem and some results on existence, uniqueness, regularity and so on, are published by KIKUCHI AND ODEN [34], HASLINGER, HLAVÁČEK AND NEČAS [29], BOIERI, GASTALDI AND KINDERLEHRER [7] , ECK [22]. The results given in these references are not restricted to linear elasticity and contact without friction.

In Section 2.1 an introduction to the basic results of linear elasticity as well as the notation are presented. Section 2.2 is concerned with the derivation of the contact condition. Also some draw backs of this condition are mentioned. In Section 2.3 connects results on linear elasticity and the contact condition to the classical formulation, which is a system of partial differential equations and boundary conditions.

2.1 Basics of Linear Elasticity

This section is far away from being complete and mathematical correct. For a more detailed description refer to CIARLET [17]. Let a domain $\Omega \subset \mathbb{R}^d$ represent the *reference configuration* of a material body, where $d \in \{2, 3\}$ is the dimension of the space. There are at least two possibilities to describe the deformation (motion)

- Let's characterise the deformation (motion) of the body by the mapping

$$\begin{aligned} &P : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}^d \\ &x = P(X, t) \qquad\qquad X \in \Omega,\, t \in \mathbb{R}_0^+ \,. \end{aligned} \tag{2.1}$$

  Here it was expected that $X = P(X, 0)$, P is injective (in the first variable), sufficiently smooth and orientation preserving. That P is orientation preserving can be expressed by the pointwise inequality

$$J(X,t) := \det\left(\frac{\partial P}{\partial X}\right)(X,t) > 0 \qquad \forall\, X \in \Omega\, \forall\, t \in \mathbb{R}_0^+ \,. \tag{2.2}$$

This kind of representation is called *material-* or *lagrangian- representation.* $P(X,t)$ describes the position of the material point X in the reference configuration, at the time t.

- Let $x \in \mathbb{R}^d$, then the deformation (motion) of the body can be described with the mapping

$$\begin{array}{ll} p : D(t) \times \mathbb{R}_0^+ \to \Omega & D(t) \subset \mathbb{R}^d \\ X = p(x,t) & \forall\, t \in \mathbb{R}_0^+, \forall\, x \in D(t)\,. \end{array} \tag{2.3}$$

  The domain $D(t)$ usually depends on time ($D(t) = P(\Omega, t)$). Without loss of generality it's assumed that $x = p(x,0)$. In this representation any point x in the domain $D(t)$ at time t is mapped back to the reference position X. This representation is called *spatial-* or *euler- representation.*

Both the *euler-* and the *lagrange-* descriptions are equivalent.

$$P(p(x,t),t) = x \qquad p(P(X,t),t) = X$$

In the following capital letters denote the *material-* or *lagrange- representation* and the small ones the *spatial-* or *euler- representation.*
In most applications the displacement of a particle X at time t given by

$$\begin{array}{l} U(X,t) = x - X = P(X,t) - X \\ u(x,t) \;\; = x - X = x - p(x,t) \end{array} \tag{2.4}$$

is of much more interest than the deformation itself.
From physics there are three important (conservation) laws which are

- Conservation of mass:

$$\frac{d}{dt}\int\limits_{P(A,t)} \rho(x,t)\,dx \;=\; 0 \qquad \forall\, A \subset \Omega\, \forall\, t \in \mathbb{R}_0^+, \tag{2.5}$$

  where ρ is the density of mass.

- Impulse equation (Newton's law):

$$\int\limits_{P(A,t)} f(x,t)\,dx + \int\limits_{\partial P(A,t)} \hat{t}(x,t,\nu)\;da(x) = \frac{d}{dt}\int\limits_{P(A,t)} \rho(x,t)v(x,t)\,dx \tag{2.6}$$

  Here $f(x,t)$ is the density of volume forces, ν the out warding normal vector (on $\partial P(A,t)$) and $v(x,t)$ the speed of the mass point $p(x,t)$.

$$v(x,t) = V(p(x,t),t) = \left(\frac{\partial P(X,t)}{\partial t}\right)(p(x,t),t) \tag{2.7}$$

- Torque equation:

$$\int_{P(A,t)} x \times f(x,t)\,dx + \int_{\partial P(A,t)} (x \times \hat{t}(x,t,\nu))\,da(x) = \frac{d}{dt}\int_{P(A,t)} x \times \rho(x,t)v(x,t)\,dx \tag{2.8}$$

  Here $\times$ denotes the vector product.
  Note: Because of Newton's law equation (2.6) is invariant under translations of the origin.

The fundamental Axiom (Euler, Cauchy) is the existence of a stress field $\hat{t}(x,t,\nu)$ which fulfils (2.6), (2.8) and some boundary conditions on $\Gamma_N \subset \partial\Omega$

$$\hat{t}(x,t,\nu_x) = l(x,t) \qquad \forall\, x \in P(\Gamma_N,t)\,\forall\, t \in \mathbb{R}_0^+ . \tag{2.9}$$

$l(x,t)$ is the surface load. From conservation of mass (2.5) the continuity equation

$$\frac{\partial\rho}{\partial t}(x,t) + \mathrm{div}_x\,(\rho(x,t)v(x,t)) = \frac{D\rho}{Dt}(x,t) + \rho(x,t)\mathrm{div}_x v(x,t) = 0 \tag{2.10}$$

can be deduced, where

$$\frac{Dy}{Dt}(x,t) = \frac{\partial y}{\partial t}(x,t) + \langle \mathrm{grad}_x y, v\rangle_{l_2(\mathbb{R}^d)}(x,t) \tag{2.11}$$

is the material derivative.
As a consequence of Newton's law (2.6) the stress field $\hat{t}(x,t,\nu)$ is linear in ν. Thus the stress field can be written as $\hat{t}(x,t,\nu) = \hat{t}(x,t)\nu$, where $\hat{t}(x,t)$ is a tensor of second order called *Cauchy stress tensor*. Using additionally the continuity equation (2.10) and *Reynolds transport theorem* the following equation can be deduced

$$\begin{aligned} \mathrm{div}_x\hat{t}(x,t) + f(x,t) &= \rho(x,t)\frac{Dv}{Dt}(x,t) && \forall\, t \in \mathbb{R}_0^+\,\forall\, x \in P(\Omega,t) \\ \hat{t}(x,t)\nu_x &= l(x,t) && \forall\, t \in \mathbb{R}_0^+\,\forall\, x \in P(\Gamma_N,t) \end{aligned} \tag{2.12}$$

As a consequence of the torque equation (2.8) the *Cauchy stress tensor* is symmetric, i.e.

$$\hat{t}(x,t) = \hat{t}^T(x,t) \qquad \forall\, t \in \mathbb{R}_0^+\,\forall\, x \in P(\Omega,t)\,. \tag{2.13}$$

Additionally the displacement $u(x,t)$ is fixed on $x \in P(\Gamma_D,t) \subset \partial P(\Omega,t)$. Summing up all results the following system of partial differential equations in *spatial*- or *euler-representation* follows.

$$\begin{aligned} \mathrm{div}_x\hat{t}(x,t) + f(x,t) &= \rho(x,t)\frac{Dv}{Dt}(x,t) && \forall\, t \in \mathbb{R}_0^+\,\forall\, x \in P(\Omega,t) \\ \hat{t}(x,t)\nu_x &= l(x,t) && \forall\, t \in \mathbb{R}_0^+\,\forall\, x \in P(\Gamma_N,t) \\ u(x,t) &= u_0(x,t) && \forall\, t \in \mathbb{R}_0^+\,\forall\, x \in P(\Gamma_D,t) \\ \hat{t}(x,t) &= \hat{t}^T(x,t) && \forall\, t \in \mathbb{R}_0^+\,\forall\, x \in P(\Omega,t) \end{aligned} \tag{2.14}$$

The next step is to transform the system of partial differential equations (2.14) given in *spatial-* or *euler- representation* into a system given in *material-* or *lagrange- representation.* This is done by using the *Piola-transformation.* Rewriting the system of partial differential equations (2.14) in *material-* or *lagrange- coordinates* results in

$$\begin{aligned} \mathrm{div}_X T(x,t) + \hat{F}(X,t) &= \rho_0(X)\frac{\partial^2 U}{\partial t^2}(X,t) && \forall\, t \in \mathbb{R}_0^+ \, \forall\, X \in \Omega\,, \\ T(X,t)\nu_X &= L(X,t) && \forall\, t \in \mathbb{R}_0^+ \, \forall\, X \in \Gamma_N\,, \\ U(X,t) &= U_0(X,t) && \forall\, t \in \mathbb{R}_0^+ \, \forall\, X \in \Gamma_D\,, \\ T(X,t)F(X,t)^T &= F(X,t)T^T(X,t) && \forall\, t \in \mathbb{R}_0^+ \, \forall\, X \in \Omega\,. \end{aligned} \tag{2.15}$$

In equation (2.15) $T(X,t)$ denotes the first *Piola-Kirchhoff* stress tensor, given by

$$T(X,t) = J(X,t)\hat{t}\left(P(X,t),t\right) F(X,t)^{-T}\,, \tag{2.16}$$

ρ_0 the density of the reference configuration Ω, $\hat{F}(X,t)$ the volume force density, given by

$$\hat{F}(X,t) = J(X,t)f(P(X,t),t)$$

and $L(X,t)$ the surface load

$$L(X,t) = l(P(X,t),t)\|J(X,t)F(X,t)^{-T}\nu_x(P(X,t),t)\|_{l_2(\mathbb{R}^d)}\,.$$

Here $F(X,t)$ denotes the deformation gradient

$$F(X,t) := \frac{\partial P}{\partial X}(X,t)\, n. \tag{2.17}$$

To solve this system of partial differential equations some more information is needed, namely information about the behaviour of the material (material law's). Assuming that the material is an elastic material and assuming the simplest case, a linear elastic material, then the material law is given via

$$\left(F^{-1}T\right)_{ij}(X,t) \;=\; \sum_{k,l=1}^{d} a_{ijkl}(X,t)E_{kl}(X,t)\,, \tag{2.18}$$

with the elasticity tensor (Hooks' tensor) $a_{ijkl}(X,t)$ and the *Green-St. Venant* strain tensor

$$E = \frac{1}{2}\left(F^T F - I\right). \tag{2.19}$$

To achieve boundness and ellipticity in the following theories it's assumed that

$$\begin{aligned} |a_{ijkl}(X,t)| &\le C_0 \\ \sum_{i,j,k,l=1}^{d} a_{ijkl}(X,t)\xi_{ij}\xi_{kl} &\ge c_0 \sum_{i,j=1}^{d} \xi_{ij}^2\,. \end{aligned} \tag{2.20}$$

Because of the symmetry of the stress tensor and the property of linear elastic materials a_{ijkl} has also to be symmetric

$$a_{ijkl}(X,t) = a_{jikl}(X,t) = a_{klij}(X,t) \qquad \forall\, i,j,k,l \in \{1,\dots,d\} \tag{2.21}$$

For homogen and isotropic materials the elasticity tensor (Hooks' tensor) is given by

$$a_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{2.22}$$

λ and μ denote Lamé parameters and both are larger than zero. In technical application's the Young-Modul and the Poison-number are more familiar than the Lamé constants. The link between these coefficients are given in the following equation.

$$\begin{aligned} \lambda &= \frac{E\nu}{(1+\nu)(1-2\nu)} \\ \mu &= \frac{E}{2(1+\nu)}\,, \end{aligned} \tag{2.23}$$

where $E > 0$ is the Young-Modul (Elastizitätsmodul) and $\nu \in (0, \frac{1}{2})$ is the Poison-number (Querkontraktionszahl).

Note that (2.18) is linear in E but not in the displacements U. Usually one is interested in a system of partial differential equations which is linear in the displacements.

This can be achieved by assuming only very small deformations, then the equations (2.15) with material law (2.18) can be linearised. This was done in CIARLET [17], who proved that the operator

$$\mathcal{A} : u \rightarrow \begin{pmatrix} -\text{div}\ ((I + \text{grad}u)(F^{-1}T)(E)) \\ (I + \text{grad}u)(F^{-1}T)(E)\nu|_{\Gamma_N} \end{pmatrix}$$

is Frechet-differentiable at $u = 0$ and

$$\mathcal{A}(0)u = \begin{pmatrix} -\text{div}\,\sigma(u) \\ \sigma(u)\nu|_{\Gamma_N} \end{pmatrix}.$$

σ denotes the linearised stress tensor

$$\sigma_{ij} = a_{ijkl}\epsilon_{kl}\,, \tag{2.24}$$

with the linearised *Green-St. Venant* strain tensor

$$\epsilon(u) = \frac{1}{2}\left(\text{grad}u + (\text{grad}u)^T\right). \tag{2.25}$$

Collecting all the results the following equations can be deduced:

$$\begin{aligned} \text{div}\,\sigma(X,t) + \hat{F}(X,t) &= \rho_0 \frac{\partial^2 U}{\partial t^2}(X,t) && \forall\, t \in \mathbb{R}_0^+\ \forall\, X \in \Omega \\ \sigma(X,t)\nu_X &= L(X,t) && \forall\, t \in \mathbb{R}_0^+\ \forall\, X \in \Gamma_N \\ u(X) &= 0 && \forall\, t \in \mathbb{R}_0^+\ \forall\, X \in \Gamma_D\,, \end{aligned} \tag{2.26}$$

where σ is given by (2.24). It can easily be seen that (2.26) is a system of partial differential equations of second order. In the special case of an isotropic body, for which (2.22) is valid, the left hand side of (2.26) has a relative simple form:

$$\text{div}\,\sigma = (\lambda + \mu)\text{grad div}u + \mu\Delta u\,.$$

2.2 Contact Condition

In the following only time independent problems are considered and thus the time derivative and the time variable is omitted. The main task in this section is to formulate the contact conditions. On the one hand the contact condition should prevent penetration of the two bodies and on the other it should describe the transfer of forces in a correct way. In the following only *material-* or *lagrange- coordinates* are used and thus it won't be distinguished between capital letters and small ones, as in the section before.

Let the bodies occupy bounded domains Ω^1, $\Omega^2 \subset \mathbb{R}^d$ (d is the dimension of the space) with Lipschitz boundaries, let $u(X)$ be the displacement field in *material-* or *lagrangian-* notation $\Omega := \Omega^1 \cup \Omega^2$, and assume that the boundaries of the domains are split into three parts,

$$\begin{aligned} \partial\Omega^1 &= \Gamma^1 = \overline{\Gamma^1_D} \cup \overline{\Gamma^1_N} \cup \overline{\Gamma^1_C} \\ \partial\Omega^2 &= \Gamma^2 = \overline{\Gamma^2_D} \cup \overline{\Gamma^2_N} \cup \overline{\Gamma^2_C}\,, \end{aligned}$$

which are open and disjoint. Only Γ^1_C and Γ^2_C may come into contact. See Figure 2.1. Further it's assumed that the body $\Omega^1 \cup \Omega^2$ is fixed by its part $\Gamma_D := \Gamma^1_D \cup \Gamma^2_D$,

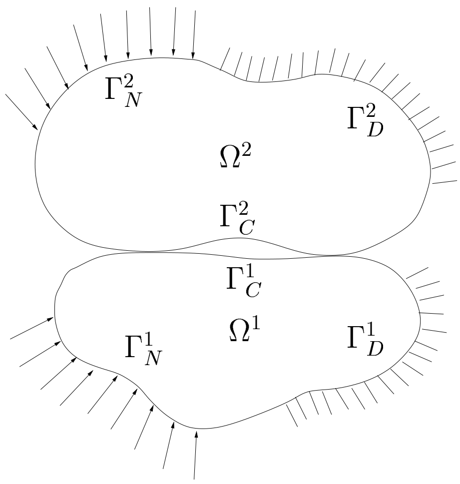


Figure 2.1: two-body contact

$$u = 0 \quad \text{on } \Gamma_D \subset \partial\Omega\,. \tag{2.27}$$

On $\Gamma_N := \Gamma^1_N \cup \Gamma^2_N$ the surface load is given, that is,

$$\sigma^{(\nu)} = L \quad \text{on } \Gamma_N \subset \partial\Omega\,, \tag{2.28}$$

where ν denotes the outer unit normal to $\partial\Omega$, L is the surface load and $\sigma^{(\nu)} := \sigma \cdot \nu$.

In Haslinger, Hlaváček and Nečas [29] one more boundary condition is presented, which represents a glide bearing. This formulation of the contact and the following theory isn't restricted to the decomposition of the boundary given above. It's also possible to handle more complicated boundary conditions as long as Γ_C^i, which is some kind of a-priori information about the region of the two bodies that may come into contact, are known.

2.2.1 Non-penetration Condition

A mathematically exact formulation of the non-penetration condition would be to define the non-convex set of admissible displacements

$$\mathcal{K} = \left\{ v \in V \ \mid v(\overset{\circ}{\Omega}_1) \cap v(\overset{\circ}{\Omega}_2) = \emptyset \right\} .$$

For this set it's not possible to handle it numerically. To find a suitable convex approximation let $X^{(i)} : \Gamma_C \subset \mathbb{R}^{d-1} \to \Gamma_C^i$ be two one to one maps with $X^{(i)} \in C^1(\Gamma_C, \Gamma_C^i)$ (See Figure 2.2). Further more define

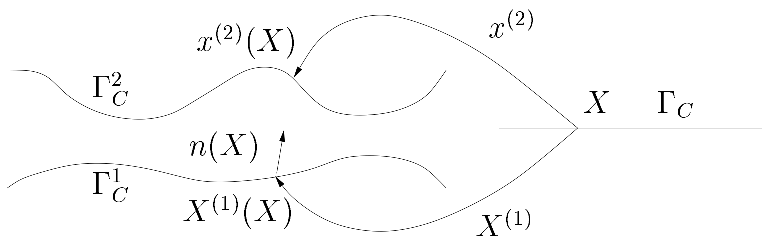


Figure 2.2: common parametrisation

$$\begin{aligned} n(X) &:= \frac{X^{(2)}(X) - X^{(1)}(X)}{|X^{(2)}(X) - X^{(1)}(X)|} \\ g(X) &:= |X^{(2)}(X) - X^{(1)}(X)| \\ u^R(X) &:= u^1 \circ X^{(1)}(X) - u^2 \circ X^{(2)}(X) \\ v_N(X) &:= \langle v(X), n(X) \rangle_{l_2(\mathbb{R}^d)} \\ v_T(X) &:= v(X) - v_N(X) n(X) , \end{aligned} \tag{2.29}$$

where $u^j = u|_{\partial\Omega^j}$ is the trace of the boundary of Ω^j. Suppose that the final contact region may be represented implicitly by the function $\psi : \mathbb{R}^d \to \mathbb{R}$ and the relation

$$\psi(y) = 0 , \tag{2.30}$$

with the non-penetration condition taking the form

$$\left\{ \begin{array}{lcl} \psi(\, (X^{(1)} + u^1(X^{(1)}))(X)\,) & \geq & 0 \\ \psi(\, (X^{(2)} + u^2(X^{(2)}))(X)\,) & \leq & 0 \end{array} \right. \quad \forall\, X \in \Gamma_C . \tag{2.31}$$

For deriving the system of partial differential equations for linear elasticity (Section 2.1) it was still assumed that u^j is very small and that it's possible to neglect terms of higher order than one. Additionally assume that $g(X)$ is also very small (comparable with u^j). Developing inequality (2.31) into a Taylor series at the point $X_0(X) := X^{(1)}(X) + \frac{1}{2}g(X)n(X)$ and neglecting higher order terms the non-penetration condition becomes

$$\left\{ \begin{array}{lcl} \psi \circ X_0(X) + \langle (\nabla\,\psi) \circ X_0(X), u^1 \circ X^{(1)}(X) - \frac{1}{2}g(X)n(X)\rangle_{l_2(\mathbb{R}^d)} & \geq & 0 \\ \psi \circ X_0(X) + \langle (\nabla\,\psi) \circ X_0(X), u^2 \circ X^{(2)}(X) + \frac{1}{2}g(X)n(X)\rangle_{l_2(\mathbb{R}^d)} & \leq & 0 \end{array} \right. \qquad \forall\, X \in \Gamma_C\,.$$

The rest term of the Taylor series is of order $o(\|u^1 \circ X^{(1)}(X) - \frac{1}{2}g(X)n(X)\|)$ in the first inequality and of order $o(\|u^2 \circ X^{(2)}(X) + \frac{1}{2}g(X)n(X)\|)$ in the second one and because of the smallness assumption above this term is neglect able. Subtracting both inequalities gives

$$\langle (\nabla\,\psi) \circ X_0(X), u^1 \circ X^{(1)}(X) - u^2 \circ X^{(2)}(X) - g(X)n(X)\rangle_{l_2(\mathbb{R}^d)} \geq 0 \qquad \forall\, X \in \Gamma_C\,.$$

$(\nabla\,\psi) \circ X_0(X)$ is a priori not known and thus assume that

$$\frac{(\nabla\,\psi) \circ X_0(X)}{\|(\nabla\,\psi) \circ X_0(X)\|_{l_2(\mathbb{R}^d)}} \approx n(X) \qquad \forall\, X \in \Gamma_C\,, \tag{2.32}$$

which should be approximately valid for small deformations. Summing up the linearised non-penetration condition results into the *signorini-* or *geometrical contact condition.*

$$u_N^R(X,t) \;=\; \langle u^R(X,t), n(X)\rangle_{l_2(\mathbb{R}^d)} \;\leq\; g(X)\,. \tag{2.33}$$

The next Lemma 2.1 will show that the penetration of the bodies is reasonable small if the parametrisation is good enough, in the sense of preventing penetration!

Lemma 2.1. *Let $\epsilon > 0$ be a small parameter and assume*

- $|u_i(X)|,\ |\epsilon_{ij}(u)| \leq \epsilon$
- $|g(X)| < 2\epsilon \quad \forall\, X \in \Gamma_C$
- *The curvature of Γ_C^1, Γ_C^2 is bounded*
- $\forall\, X \in \Gamma_C$

$$\begin{array}{l} \|n(X) - \nu^1\left(X^{(1)}(X)\right)\|_{l_2(\mathbb{R}^d)} \leq \|\nu^1\left(X^{(1)}(X)\right) + \nu^2\left(X^{(2)}(X)\right)\|_{l_2(\mathbb{R}^d)} \\ \|n(X) + \nu^2\left(X^{(2)}(X)\right)\|_{l_2(\mathbb{R}^d)} \leq \|\nu^1\left(X^{(1)}(X)\right) + \nu^2\left(X^{(2)}(X)\right)\|_{l_2(\mathbb{R}^d)}\,, \end{array}$$

where $\nu^1\left(X^{(1)}\right), \nu^2\left(X^{(2)}\right)$ are the out warding normal vectors at the boundaries Γ_C^1, Γ_C^2.

Then the condition, that Γ_C^1 and Γ_C^2 don't intersect, is equivalent to

$$u_N^R(x) \leq g(x) + r(x),$$

where the error $r(x)$ can be estimated by

$$|r(x)| \leq K\epsilon^{3/2}\,.$$

Proof. For proof see Eck [22, Lemma 1.3]. □

Remark 2.2. It's not a-priori clear how to choose the parametrisation $X^{(j)}$. In fact it's not clear whether there are parametrisations which result into the "exact" solution of the "real" problem or not. From the physical point of view it seems to be a "good" idea to choose the parametrisation such that the gap $g(X)$ is minimal. This kind of parametrisation can result into an ill-posed problem, because it's possible that $\exists\, A \subset \Gamma_C$, $\text{meas}(A) > 0$ with $\text{meas}(X^{(j)}(A)) = 0$ and this results into a not closed admissible set of displacements. An Example is given in Carstensen, Scherf and Wriggers [15, Proposition 3.2]. Nevertheless this parametrisation is often used. An other idea is to fix only one parametrisation and the unit vector $n(X)$. Then it's possible to reconstruct the second parametrisation. It's also possible that this choice results into an ill-posed problem (same reason as above).

Remark 2.3. One draw back is that the non-penetration condition is only formulated for small deformations. In mechanics there are a lot of more realistic material laws, which allow to handle big deformations as well, but there is no non-penetration condition for this case. The first idea to get ride of this problem may be choose the parametrisation depending on the displacement and applying some fix point iteration. This was done by the author, but the fixed point iteration resulted into a "oscillating" sequence. One idea to prevent this is to use the active restrictions of the last two steps. But this doesn't fit into the algorithm presented in this thesis and thus it wasn't tried.
An other idea is to consider the problem of big deformations as a quasi-static problem. This means that all forces and surface loads are applied arbitrary slowly. In every step it's now possible to solve a problem with small deformations (if there are enough steps to apply the forces), which fits into the non-penetration condition above. One more advantage is that it seems to be very easy to implement, not depending on the solving algorithm of the stationary contact problem. Nevertheless this wasn't implemented for this thesis.

2.2.2 Transfer of Forces

It was mentioned before that the contact condition shouldn't only prevent penetration, it also should describe the transfer of forces at the boundaries. This section isn't mathematically correct, it's only an approximate approach. For a mathematical correct modelling a process similar to the derivation of the linear elasticity (Section 2.1) has do be done.

- As a direct consequence of Newton's law the normal components (normal to the deformed configuration) of the stress tensor, in *spatial-* or *euler- notation*, have to be equal, if two points are in contact ($\sigma^{(\nu^1)}(x) = \sigma^{(\nu^2)}(x)$). Because of our model a point is in contact iff $u_N^R(X) = g(X)$.
  Note: The physical picture of contact isn't the same as the *signorini-* or *geometric-contact condition*. In the physical picture two points are in contact iff the points coincide. In the *signorini-* or *geometric- picture* two points are in contact iff the points are represented with one point X in the parameter domain Γ_C via the parametrisation

$X^{(j)}(X)$ and if they are connected with a hypersurface, defined via $n(X)$. It's only to hope that for a "good" parametrisation both pictures coincide.

This condition in *spatial-* or *euler-notation* can be transformed by the *Piola transformation* (as in Section 2.1) into *material-* or *lagrange- formulation* and looks as follows

$$\sigma(X) := \sigma^{(\nu^1)}\left(X^{(1)}(X)\right) J_1(X) = -\sigma^{(\nu^2)}\left(X^{(2)}(X)\right) J_2(X) \qquad \forall\, X \in \Gamma_C, \tag{2.34}$$

where $J_k = \left|\det\left\langle \frac{\partial X^{(k)}(X)}{\partial X_i}, \frac{\partial X^{(k)}(X)}{\partial X_j}\right\rangle_{l_2(\mathbb{R}^d)}\right|$ is the gram determinant, and ν^j are the out warding normals on the reference configuration. The gram determinants J_k are consequences of the transformation rule of surfaces (transformation of $\Gamma_C^k \to \Gamma_C$). Also for this derivation the smallness assumption of the displacements was used (similar to linear elasticity Section 2.1).

- Because only compressive forces can act, it's possible to deduce that $\langle \sigma\nu^k, \nu^k\rangle_{l_2(\mathbb{R}^d)} \leq 0$. (in the *spatial-* or *euler- notation*, where ν^j are the out warding normals of the deformed configuration) Transforming this condition into *material-* or *lagrangian-notation* and using assumption (2.32) the inequality

$$\sigma_N(X) = (-1)^j J_j(X)\langle \sigma^{(\nu^j)} \circ X^{(j)}, n\rangle_{l_2(\mathbb{R}^d)}(X) \leq 0 \qquad \forall\, X \in \Gamma_C\, \forall\, j \in \{1, 2\}$$

  follows, where n is the vector given in (2.29) and $\nu^{(j)}$ is the out warding normal on the reference configuration Γ_C^j. σ_N coincides with the notation given in (2.29).

- Forces can only be transfered at contact zones. Here it's necessary to distinguish between the physical and the "geometric" picture. Contact occurs iff $u_N^R(X) - g(X) = 0$.

$$\Longrightarrow \quad \sigma_N(X) = 0 \quad \text{if} \quad u_N^R(X) - g(X) < 0\,.$$

- Because frictionless contact is assumed no tangential forces can be transfered. This is in *material-* or *lagrangian- notation*, with the same approximations as above,

$$\sigma_T(X) = 0 \qquad \forall\, X \in \Gamma_C\,, \tag{2.35}$$

  where the notation introduced in (2.29) was used.

Summarising all these results, the contact condition follows.

$$\left.\begin{array}{lcl} \left(\sigma^{(\nu^1)} \circ X^{(1)}\right)(X) J_1(X) & = & -\left(\sigma^{(\nu^2)} \circ X^{(2)}\right)(X) J_2(X) \\ u_N^R(X) & \leq & g(X) \\ \sigma_N(X) & \leq & 0 \\ \sigma_T(X) & = & 0 \\ \sigma_N(X)\left(u_N^R(X) - g(X)\right) & = & 0 \end{array}\right\} \quad \forall\, X \in \Gamma_C \tag{2.36}$$

For a contact formulation with friction refer to KIKUCHI AND ODEN [34], HASLINGER, HLAVÁČEK AND NEČAS [29], ECK [22].

2.3 Classical Formulation of the Body-Body Contact Problem without friction

The *contact condition* (2.36) and the partial differential equation (2.26) result into the classical formulation of the *linearised body-body contact problem.*

$$
\begin{array}{lll}
-\operatorname{div}\sigma(X,\nabla u) & = \hat{F}(X) & \text{in } \Omega \\
u(X) & = 0 & \text{on } \Gamma_D \\
\sigma^{(\nu^j)}(X,\nabla u) & = L_i(X) & \text{on } \Gamma_N \\
\left(\sigma^{(\nu^1)}(.,\nabla u)\circ X^{(1)}\right)(X)J_1(X) & = -\left(\sigma^{(\nu^2)}(.,\nabla u)\circ X^{(2)}\right)(X)J_2(X) & \text{on } \Gamma_C \\
\left.\begin{array}{ll}
u_N^R(X) & \le\ g(X) \\
\sigma_N(X,\nabla u) & \le\ 0 \\
\sigma_T(X,\nabla u) & =\ 0 \\
\sigma_N(X,\nabla u)\left(u_N^R(X)-g(X)\right) & =\ 0
\end{array}\right\} & & \text{on } \Gamma_C\,,
\end{array}
\tag{2.37}
$$

where $\sigma(X,\nabla u)$ is given by (2.24) and $X^{(1)}, X^{(2)}$ are the parametrisations from above.

Remark 2.4. This model is approximately valid if the displacements are small, not due to the material law, no because of the contact condition. Note that the solution depends on the choice of the parametrisation and it's not clear how to choose it, to get a solution which may also be a realistic one. An other problem is that if the displacement field is known, which fulfils the system of partial differential equations above and fulfils a realistic contact condition (physical non-penetration condition and transfer of forces) and if the parametrisation is constructed in an obvious way, it's not clear that this displacement field is also a solution of the contact problem with constructed parametrisation. All in all, the *signorini-* or *geometrical- contact condition* seems not to be very satisfying but the author don't know a better one and thus this condition is used in this thesis.

Chapter 3

Sobolev Spaces

For numerical analysis of partial differential equations (PDE) usually *Sobolev spaces* are used. On the one hand these spaces allow a more general solution of the PDE and on the other the analysis of existence and uniqueness is usually much easier. Additionally *Sobolev spaces* allows finite element (FE) approximations of the solutions and thus a simple discretisation of the PDE.

An introduction to Sobolev spaces can be found in Adams [1] or Lions and Magenes [39]. In this thesis it's necessary to define positive functions on *Sobolev spaces*. This set of positive functions should be represented by a condition on the dual space. Thus a partial ordering is needed. For this refer to Kikuchi and Oden [34] and references in there. In numerical analysis scaling arguments are used, i.e. that results proved on *Sobolev spaces* of integer type are extended to the corresponding result on *Sobolec spaces* of fractional order. This scaling argument is possible due to the fact that *Sobolev spaces* of fractional order can be represented as interpolation spaces. For interpolation spaces and properties of this spaces refer to Adams [1], Lions and Magenes [39] or Bramble [10].

In Section 3.1 *Sobolev spaces* and some of their important properties are presented. Especially the embedding results, the possibility to define traces and the validity of the *Green's formula* are needed in this thesis. It was mentioned above that *Sobolev spaces* of fractional order can be represented as interpolation spaces. This fact is presented in Section 3.2. In Section 3.3 an abstract partial ordering on vector spaces is given and then this abstract ordering is applied to *Sobolev spaces* to define positive functions for *Sobolev spaces*. The important property of the partial ordering is the possible representation of *cones* by it's *polar cones*. Finally an abstract result for equivalent norms on *Banach spaces* is presented in Section 3.4 which enables the prove of V-ellipticity for variational inequalities.

3.1 Preliminary Results and Definitions

Natural spaces for variational problems are *Sobolev spaces*. The only which *Sobolev spaces* are needed in this thesis are *Hilbert* type spaces.

Let Ω be an open and bounded domain of $\mathbb{R}^d$. To avoid technical difficulties assume that Ω is polygonal. Assume that Γ_D is open subset of $\partial\Omega$. Let $\omega \subseteq \Omega$ be an open Lipschitz continuous boundary. The space $C^\infty(\omega)$ denotes the space of infinitely differentiable functions on ω and the subspace $C^\infty_{0,D}(\omega)$ consists of all functions of $C^\infty(\omega)$ with vanishing values of the function and all its derivatives on Γ_D.

Let $\langle .,. \rangle_{L_2(\omega)\times L_2(\omega)}$ be the inner product with the associated norm

$$\|v\|^2_{0,\omega} := \langle v, v\rangle_{L_2(\omega)\times L_2(\omega)}$$

on $L_2(\omega)$.

For $k \in \mathbb{N}$ the following norm is defined recursively

$$\|v\|_{k,\omega} := \left(\|v\|^2_{k-1,\omega} + \sum_{i=1}^{d} \left\|\frac{\partial v}{\partial x_i}\right\|^2_{k-1,\omega}\right)^{\frac{1}{2}}. \tag{3.1}$$

Let $\sigma \in]0,1[$, then define the semi norm

$$|v|_{\sigma,\omega} := \int\limits_{\omega\times\omega} \frac{|v(x)-v(y)|^2}{|x-y|^{d+2\sigma}}\, d(x\times y)\,. \tag{3.2}$$

With this it's possible to define for all $\alpha \geq 0$ the norm

$$\|v\|^2_{\alpha,\omega} := \|v\|^2_{k,\omega} + \|v\|^2_{\sigma,\omega}\,, \tag{3.3}$$

where $k \in \mathbb{N}$, $\sigma \in]0,1[$ such that $\alpha = k + \sigma$.

Definition 3.1 (Sobolev spaces). *Let $\alpha \geq 0$ and $\omega \subseteq \Omega$ an* Lipschitz *bounded domain, then define the following* Sobolev spaces *of order α*

$$H^\alpha(\omega) := \overline{C^\infty(\overline{\omega})}^{\|\cdot\|_{\alpha,\omega}} \tag{3.4}$$

$$H_0^{-\alpha}(\omega) := (H^\alpha(\omega))' \tag{3.5}$$

$$H^\alpha_{0,D}(\omega) := \overline{C^\infty_{0,D}(\overline{\omega})}^{\|\cdot\|_{\alpha,\omega}} \tag{3.6}$$

$$H^{-\alpha}(\omega) := (H^\alpha_0(\omega))'\,. \tag{3.7}$$

In numerical analysis *Green's formula* for *Sobolev spaces* is essential, also in this thesis. For this the space $H(\mathrm{div},\Omega)$ has to be introduced.

Definition 3.2. *The space $H(div,\Omega)$ is defined by*

$$H(div,\Omega) := \{\, q \in (L_2(\Omega))^d \mid div\, q \in L_2(\Omega)\,\}\,, \tag{3.8}$$

with the norm

$$\|q\|^2_{div,\Omega} := \|q\|^2_{0,\Omega} + \|div\, q\|_{0,\Omega}\,. \tag{3.9}$$

For this thesis the space $H_{00}^{\frac{1}{2}}(\Gamma)$, with $\Gamma \subset \partial\Omega$ is important.

Definition 3.3. *The space $H_{00}^{k+\frac{1}{2}}(\Gamma)$ is defined by*

$$H_{00}^{k+\frac{1}{2}}(\Gamma) := \{\, v \in H^{k+\frac{1}{2}}(\Gamma) \,|\, \rho^{-\frac{1}{2}} D^\alpha v \in L_2(\Gamma) \quad \forall\, |\alpha| = k \,\} \tag{3.10}$$

with $\rho \in C^\infty(\overline{\Gamma})$, $\rho > 0$ in Γ and vanishing on Γ of the order $d(x,\Gamma)$, i.e.

$$\lim_{x \to x_0} \frac{\rho(x)}{d(x,\Gamma)} = d \neq 0 \qquad \forall\, x_0 \in \Gamma\,.$$

The norm on $H_{00}^{k+\frac{1}{2}}(\Gamma)$ is defined by

$$\|v\|^2_{H_{00}^{k+\frac{1}{2}}(\Gamma)} = \|v\|^2_{H^{k+\frac{1}{2}}(\Gamma)} + \sum_{|\alpha|=k} \|\rho^{-\frac{1}{2}} D^\alpha v\|_{L_2(\Gamma)} \tag{3.11}$$

Note that these *Sobolev spaces* are *Hilbert spaces.* One of the important properties of *Sobolev spaces* is that they can be embedded in several other spaces. In some cases this embedding is even compact. The most important compact embeddings for this thesis are denoted in the famous *Rellich-Kandrachov* theorem.

Theorem 3.4 (Rellich-Kondrachov theorem). *Let $\Omega \subset \mathbb{R}^d$, $\Gamma = \partial\Omega \in C^{0,1}$ and $j \in \{0, ..., k\}$ with $k \in \mathbb{N}$. Then there holds the following compact embedding:*

$$\begin{array}{lll} H^k(\Omega)\ (\textit{resp. } H_0^k(\Omega)) \hookrightarrow^c H^j(\Omega)\ (\textit{resp. } H_0^j(\Omega)) & & \textit{for } j < k \\ H^k(\Omega) \hookrightarrow^c C^j(\overline{\Omega}) & & \textit{for } k - \frac{d}{2} > j \end{array} \tag{3.12}$$

Proof. see Adams [1, Theorem 6.2] □

For PDEs it's necessary to have boundary values, but *Sobolev spaces* are subspaces of L_2 and in L_2 boundary values make no sense. Nevertheless it's possible to define boundary values for *Sobolev spaces* as long as the order is large enough. The next theorem is a special case of a more general trace theorem and guarantees that traces are well defined for *Sobolev spaces* which are needed in this thesis.

Theorem 3.5. *Let $\Omega \in \mathcal{C}^{0,1}$, $\Gamma \subseteq \partial\Omega$ and let γ_Γ be the operator defined by*

$$\gamma_\Gamma(v) = v|_\Gamma \qquad \forall\, v \in C^\infty(\overline{\Omega})\,. \tag{3.13}$$

Then γ_Γ can be extended to a continuous linear operator, also denoted γ_Γ, from $H^1(\Omega)$ onto $H^{\frac{1}{2}}(\Gamma)$ and this operator is surjective with ker $\gamma_\Gamma = H^1_{0,\partial\Omega/\Gamma}$.

Proof. see Adams [1, Theorem 7.53] or Lions and Magenes [39, I Theorem 8.3] □

Remark 3.6. If the trace of the space $H^1_{0,D}(\Omega)$ is considered with $\Gamma = \mathrm{int}\partial\Omega/\Gamma_D$ then the trace operator isn't surjective onto $H^{\frac{1}{2}}(\Gamma)$. The correct space will be the space $H_{00}^{\frac{1}{2}}(\Gamma)$. For this space the same results as before are valid (at least if $\Omega \in \mathcal{C}^\infty$).

Remark 3.7. A consequence of this surjectivity (indeed of the existence of a continuous extension operator) and the trace operator γ_Γ is that

$$\|g\|_{\frac{1}{2},\Gamma} \equiv \inf_{\substack{v\in H^1(\Omega)\\ \gamma_\Gamma(v)=g}} \|v\|_{1,\Omega} \tag{3.14}$$

The last property of *Sobolev spaces* which will be needed is the abstract *Green's formula.*

Lemma 3.8. *For $q \in H(div,\Omega)$ the scalar product $\langle q,n\rangle_{l_2(\mathbb{R}^d)}|_\Gamma \in H^{-\frac{1}{2}}(\Gamma)$ can be defined and the* Green's formula *is valid.*

$$\langle div\, q, v\rangle_{0,\Omega} + \langle q, grad\, v\rangle_{0,\Omega} = \langle \langle q,n\rangle_{l_2(\mathbb{R}^d)}, v\rangle_{H^{-\frac{1}{2}}(\Gamma)\times H^{\frac{1}{2}}(\Gamma)} \qquad \forall\, v \in H^1(\Omega) \tag{3.15}$$

Proof. see Brezzi and Fortin [12, III Lemma 1.1] □

Theorem 3.9. *The trace operator*

$$\langle .,n\rangle_{l_2(\mathbb{R}^d)}|_\Gamma : H(div,\Omega) \to H^{-\frac{1}{2}}(\Gamma) \quad q \mapsto \langle .,n\rangle_{l_2(\mathbb{R}^d)}|_\Gamma \tag{3.16}$$

is surjective.

Proof. see Brezzi and Fortin [12, III Lemma 1.2] □

3.2 Interpolation Spaces

Important for this thesis is that *Sobolev spaces* of fractional order can be represented as interpolation spaces. For this short introduction the real method of interpolation is used.

Let $(X, \langle\, . \,,\, . \,\rangle_{X\times X})$ and $(Y, \langle\, . \,,\, . \,\rangle_{Y\times Y})$ be two *Hilbert spaces* with embedding $Y \rightharpoonup X$ and Y dense in X. For $t > 0$ define the K-functional as

$$K(t,u)^2 := \inf_{\substack{x\in X,\, y\in Y\\ u=x+y}} \|x\|_X^2 + t^2\|y\|_Y^2\,. \tag{3.17}$$

The interpolation norm $\|\,.\,\|_{[X,Y]_\theta}$ for $\theta \in]0,1[$ is defined by

$$\|u\|_{[X,Y]_\theta}^2 := \int_0^\infty t^{-1-2\theta} K(t,u)^2\, dt\,. \tag{3.18}$$

The interpolation norm fulfils the parallelogram law and thus the interpolation space $([X,Y]_\theta = \overline{Y}^{\|\,.\,\|_{[X,Y]_\theta}}$ is a *Hilbert space.*

Remark 3.10. In Bramble [10, Theorem B.2.] it was proved that the real method of interpolation is equivalent to the introduction of interpolation spaces by spectral methods, i.e. by defining $[X,Y]_\theta := D(\Lambda^{1-\theta})$, where $D(\Lambda^{1-\theta})$ denotes the domain of the operator Λ and Λ is such that $\|\,.\,\|_X^2 \simeq \|\,.\,\|_Y^2 + \|\Lambda\,.\,\|_Y^2$ (see Lions and Magenes [39, I Definition 2.1]).

The next result proves that linear, bounded operators, defined on both spaces X, Y are bounded on the interpolation space $[X,Y]_\theta$. This result is often used as scaling argument in the FE analysis.

Theorem 3.11. *Let X, Y and $\mathcal{X}$, $\mathcal{Y}$ be as above, and L be a linear operator, $L \in \mathcal{L}(X,\mathcal{X}) \cap \mathcal{L}(Y,\mathcal{Y})$ with norm bounds*

$$\begin{aligned} \|Lv\|_{\mathcal{X}} &\leq c_X \|v\|_X \qquad \forall\, v \in X\,, \\ \|Lv\|_{\mathcal{Y}} &\leq c_Y \|v\|_Y \qquad \forall\, v \in Y\,. \end{aligned}$$

Then for $\forall\, \theta \in]0,1[$: $\quad L \in \mathcal{L}([X,Y]_\theta, [\mathcal{X},\mathcal{Y}]_\theta)$ with spectral bound

$$\|Lv\|_{[\mathcal{X},\mathcal{Y}]_\theta} \;\leq\; c_X^{1-\theta} c_y^{\theta} \|v\|_{[X,Y]_\theta}\,. \tag{3.19}$$

Proof. see Lions and Magenes [39, I Theorem 5.1], Bramble [10, Theorem B.4.] □

In this thesis also the characterisation of the dual space of $[X,Y]_\theta$ is needed. Because $Y \rightharpoonup X$ and Y is dense in X, the converse is valid for its dual, i.e. $X' \rightharpoonup Y'$ and X' is dense in Y'. The next theorem guarantees what is expected.

Theorem 3.12.

$$[X,Y]'_\theta \;=\; [Y',X']_{1-\theta} \qquad \forall\, \theta \in]0,1[\tag{3.20}$$

Proof. see Lions and Magenes [39, I Theorem 6.2] □

It was mentioned above that interpolation spaces are needed in this thesis because *Sobolev spaces* of fractional order can be represented as interpolation space of *Sobolev spaces* with integer order.

Theorem 3.13. *Let Ω be a bounded domain with $\partial\Omega \in C^{0,1}$. Let $s_1 > s_2 \geq 0$ and $\theta \in]0,1[$. Set $s(\theta) = (1-\theta)s_1 + \theta s_2$. Then*

$$\begin{aligned} [H^{s_1}(\Omega), H^{s_2}(\Omega)]_\theta &= H^{s(\theta)}(\Omega) & & \\ [H_0^{s_1}(\Omega), H_0^{s_2}(\Omega)]_\theta &= H_0^{s(\theta)}(\Omega) & & s_1,\, s_2\; s(\theta) \notin \mathbb{N}_0 + \tfrac{1}{2} \\ [H_0^{s_1}(\Omega), H_0^{s_2}(\Omega)]_\theta &= H_{00}^{s(\theta)}(\Omega) & & s_1,\, s_2 \notin \mathbb{N}_0 + \tfrac{1}{2}\;\; s(\theta) \in \mathbb{N}_0 + \tfrac{1}{2} \end{aligned}$$

Proof. see Lions and Magenes [39, I Theorem 9.6, Theorem 11.6, Theorem 11.7] □

3.3 Partial Ordering of Sobolev Spaces

To derive a *weak-* or *primal- formulation* of the system of partial differential inequalities (2.37) some partial order properties are needed.

First an abstract partial ordering for a *linear space* Q is introduced. Suppose that Q is a normed space (not necessary but enough for this thesis). The following definition of a *cone* is taken from Ekland and Temam [23].

Definition 3.14. *A non-empty subset* $\mathcal{C} \subseteq Q$ *is a* cone with vertex 0 *(simple* cone*) if*

$$\begin{aligned} \lambda\,\mathcal{C} \quad &\subseteq \mathcal{C} \qquad\qquad \forall\,\lambda > 0 \\ \mathcal{C} + \mathcal{C} &\subseteq \mathcal{C}\,. \end{aligned} \tag{3.21}$$

The cone *is* pointed *or* unpointed *according to whether* $0 \in \mathcal{C}$ *or* $0 \notin \mathcal{C}$.
A pointed cone with vertex 0 *is* salient *if* $\mathcal{C} \cap \{-\mathcal{C}\} = \{0\}$.

With a *pointed cone* it's possible to associate a partial ordering $\geq$ on Q by setting

$$p \geq q \quad :\Leftrightarrow \quad p - q \in \mathcal{C}\,. \tag{3.22}$$

Note the relation $\geq$ is reflexive and transitive and if the *pointed cone* is *salient*, then $\geq$ is antisymmetric and thus $\geq$ is an order relation on all of Q. Conversely, the introduction of any partial ordering $\geq$ on Q defines a *pointed cone.* Usually $\mathcal{C}$, associated with the partial ordering $\geq$, is denoted as *positive cone* $\mathcal{C}_+$ and likewise the *pointed cone* $\mathcal{C}_- = -\mathcal{C}_+$ is the *negative cone.*

The introduction of a *positive cone* $\mathcal{C}_+$ in Q makes it possible to add *inequalities* such as $q \geq 0$, $q \leq 0$ to the linear structure of Q. The partial ordering is also compatible with the structure of the linear space Q, i.e.

$$\begin{aligned} q \geq 0 \quad &\Longrightarrow \qquad \lambda\, q \geq 0 \qquad\; \forall\,\lambda \geq 0 \\ p \geq q \quad &\Longrightarrow \quad p + r \geq q + r \quad \forall\, r \in Q\,. \end{aligned}$$

For the *pointed cone* $\mathcal{C}_+$ it's possible to define the *polar cone* $\mathcal{C}_+^*$ in the dual space Q' of Q by setting

$$\mathcal{C}_+^* \;:=\; \{q^* \in Q' \;|\; \langle q^*, q\rangle_{Q'\times Q} \geq 0\;\forall\, q \in \mathcal{C}_+\}\,. \tag{3.23}$$

Note $\mathcal{C}_+^0$ is a *pointed cone* and closed even if $\mathcal{C}_+$ is not.

It's possible to repeat the game and build the polar of the *polar cone* $\mathcal{C}_+^0$. This *polar cone* is living in the space Q''. It's well known that it's possible to consider Q as a subset of Q'' and thus only the part living in Q is considered as the *polar* of the *polar cone.*

$$\mathcal{C}_+^{00} \;:=\; \left\{q \in Q \;|\; \langle q^*, q\rangle_{Q'\times Q} \geq 0\;\forall\, q^* \in \mathcal{C}_+^0\right\}. \tag{3.24}$$

The next lemma proves a relationship between $\mathcal{C}_+^{00}$ and $\mathcal{C}_+$.

Lemma 3.15. *Let* $\mathcal{C}_+$ *be a* cone*, then*

$$\mathcal{C}_+^{00} = \overline{\mathcal{C}_+}\,, \tag{3.25}$$

where $\overline{A}$ *is the closure of* A.

Proof.

$'' \supseteq ''$: Because $\mathcal{C}_+^{00}$ is closed only $\mathcal{C}_+ \subseteq \mathcal{C}_+^{00}$ has to be verified. Let $q \in \mathcal{C}_+$ then $\langle \mathcal{C}_+^0, q\rangle_{Q'\times Q} \geq 0$ and thus $q \in \mathcal{C}_+^{00}$.

$'' \subseteq ''$: Assume that $q \in \mathcal{C}_+^{00}/\overline{\mathcal{C}_+}$. As a consequence of separation of convex sets (*Hahn Banach theorem* see HEUSER [31, Satz 42.5]) there exists an element $q^* \in Q'$ and an $\alpha \in \mathbb{R}$ such that

$$\langle q^*, q \rangle_{Q' \times Q} < \alpha < \langle q^*, p \rangle_{Q' \times Q} \qquad \forall\, p \in \mathcal{C}_+$$

Because $\mathcal{C}_+$ is a *cone* and $\langle q^*, q \rangle_{Q' \times Q}$ is bounded, it follows that

$$\begin{aligned} \langle q^*, p \rangle_{Q' \times Q} &\geq 0 \qquad \forall\, p \in \mathcal{C}_+ \\ \langle q^*, q \rangle_{Q' \times Q} &< 0 \end{aligned}$$

and thus $q^* \in \mathcal{C}_+^0$ which is a contradiction to $\langle q^*, q \rangle_{Q' \times Q} < 0$. □

If $\mathcal{C}_+$ is closed then it's possible, because of Lemma 3.15, to represent $\mathcal{C}_+$ by $\mathcal{C}_+^{00}$.

Note that due to the convexity and the theorem of MAZUR (See HEUSER [31, Satz 59.4]) *weakly closed* and *closed* is equivalent.

In this thesis a partial ordering on *Sobolev spaces* is needed.

Definition 3.16. *Define the set of positive functions in the space $H^\alpha(\Gamma)$ for $\alpha \in [0,1]$ via*

$$\mathcal{C}_+ := \overline{\{\, v \in C^\infty(\Gamma) \,|\, v \geq 0 \,\}}^{\|\cdot\|_\alpha} . \tag{3.26}$$

Remark 3.17. Due to the fact (see KIKUCHI AND ODEN [34, 5 Theorem 5.2] or KINDERLEHRER AND STAMPACCHIA [35]) that

$$\max\{0, u\} \in H^\alpha(\Gamma) \qquad \forall\, u \in H^\alpha(\Gamma)\, \forall\, \alpha \in [0,1] ,$$

the set of positive functions in $H^\alpha(\Gamma)$ can be rewritten by

$$\mathcal{C}_+ := \{\, v \in H^\alpha(\Gamma) \,|\, v \geq 0 \quad \text{a.e.} \,\} .$$

Note that due to the definition $\mathcal{C}_+$ is closed and thus, due to Lemma 3.15, $\mathcal{C}_+ = \mathcal{C}_+^{00}$.

3.4 Equivalent Norms

For this thesis some results about equivalent norms for *Sobolev spaces* are needed.

Theorem 3.18. *Let $(X, \|\,.\,\|_X)$, $(Y, \|\,.\,\|_Y)$ be two* Banach spaces. *Let $X \rightharpoonup^c Y$ be a compact embedding and let $|\,.\,|_A$ be a semi-norm on X with kernel X_0. Assume that the following norms are equivalent*

$$\|\,.\,\|_X \simeq \|\,.\,\|_Y + |\,.\,|_A .$$

Then the following is true:

i. *The kernel X_0 is of finite dimension. The semi-norm $|\,.\,|_A$ is equivalent to the norm on the factor space, i.e.*

$$|u|_A \simeq \inf_{u_o \in X_0} \|u - u_0\|_X \qquad \forall\, u \in X\,.$$

ii. *Let $|\,.\,|_B$ be a continuous semi-norm on X such that there holds for all $u \in X$*

$$|u|_A + |u|_B = 0 \quad \Longrightarrow \quad u = 0$$

Then there holds the equivalence of norms

$$|\,.\,|_A + |\,.\,|_B \simeq \|\,.\,\|_X$$

iii. *Let $Z \subset X$ be a closed subspace such that $Z \cap X_0 = \emptyset$,then there holds the equivalence of norms*

$$|u|_A \simeq \|u\|_X \qquad \forall\, u \in Z\,.$$

Proof. see Girault and Raviart [25, Theorem 2.1] or Showalter [46, II Proposition 5.2] □

Remark 3.19. Choosing $X = H^{k+1}(\Omega)$, $Y = H^k(\Omega)$, with $k \in \mathbb{N}_0$ and $|\,.\,|_A = |\,.\,|_{k+1,\Omega}$. It's known that $X_0 = \text{kern}\,|\,.\,|_{k+1,\Omega} = \mathcal{P}^k(\Omega)$, where $\mathcal{P}^k(\Omega)$ is the space of polynomials of degree less than $k+1$. If the semi-norm $|\,.\,|_B$ is chosen such that

$$\forall\, v \in \mathcal{P}^k(\Omega) : \quad |v|_B = 0 \Longrightarrow u = 0\,,$$

then from Theorem 3.18 it's deduced that $\|\,.\,\|_{k+1,\Omega} \simeq |\,.\,|_B + |\,.\,|_{k+1,\Omega}$.

The *Brambel-Hilbert* lemma is just a consequence of Theorem 3.18

Lemma 3.20 (Bramble-Hilbert). *Let $k \in \mathbb{N}$ and Y be a* Hilbert space. *Let $L : H^k \to Y$ be a linear bounded operator. Assume that L vanishes on the space $\mathcal{P}^{k-1}$ of polynomials up to order $k-1$. Then L is bounded by the semi-norm, i.e.*

$$\|Lu\|_X \preceq |u|_k \qquad \forall\, u \in H^k\,.$$

Chapter 4

Variational Inequalities

A lot of practical problems, also the body-body contact problem can be formulated in a variational inequality. The spaces which are usually used are *Sobolev spaces*. The reformulation of a classical partial differential inequality into a variational inequality isn't only done for numerical calculations, especially for the analysis of variational inequalities there are powerful tools available. The aim of this chapter is to present some results for variational inequalities and applying them to the body-body contact problem.

The reformulation of the body-body contact problem into a variational inequality and the proof of the equivalence of the variational formulation and the classical formulation under certain smoothness assumptions is done in Kikuchi and Oden [34], Boieri, Gastaldi and Kinderlehrer [7] and Eck [22]. For existence and uniqueness results for variational inequalities as well as for several equivalent formulations for variational inequalities refer to Eckland and Temam [23], Showalter [46] and Kinderlehrer and Stampacchia [35]. In Kikuchi and Oden [34], Haslinger, Hlaváček and Nečas [29] and Eck [22] these abstract results for variational inequalities are applied to the body-body contact problem.

In Section 3.3 a partial ordering for Sobolev spaces was introduced. With this partial ordering it's possible to formulate the *primal-* or *weak-* formulation of the body-body contact problem. This is done in Section 4.1. Abstract existence and uniqueness results for variational inequalities are denoted in Section 4.2. In Section 4.3 several equivalent formulations for the variational inequality are presented. Especially the equivalence of *primal-* or *weak* formulation, *saddle point-* or *mixed* formulation and the *dual* formulation is presented. Finally the existence and uniqueness of the body-body contact problem under certain assumptions is proved in Section 4.4. and uniqueness result for the body-body contact problem is presented.

4.1 Variational Formulation of the Body-Body Contact Problem

To achieve a reasonable variational formulation the space of *admissible displacement functions* have to be fixed.

$$V := H^1_{0,D}(\Omega^1)^d \times H^1_{0,D}(\Omega^2)^d \tag{4.1}$$

The space V is the space of displacements without restrictions. The contact restrictions are taken into consideration by defining the set of *admissible displacements* K.

Definition 4.1. *The set of* admissible displacements K *is defined by*

$$K = \left\{ v \in V \mid v^R_N(X) - g(X) \leq 0 \quad a.e\ on\,\Gamma_C \right\} , \tag{4.2}$$

where Γ_C is the common parameter domain of $X^{(j)}$ which maps Γ_C on to Γ^j_C.

W.l.o.g. assume that $X^{(1)}$ is the identity map, i.e. $\Gamma_C = \Gamma^1_C$. It's obvious that K is a convex set, but K may not be closed. Carstensen, Scherf and Wriggers [15, Proposition 3.2] presented an example of a parametrisation, a very common parametrisation, and a domain Ω such that K isn't weakly closed and thus not closed (Mazur). With suitable assumptions on the parametrisations $X^{(j)}$ it's possible to achieve that K is closed.

Proposition 4.2. *Assume that the parametrisations $X^{(j)}$ fulfils the following condition*

$$\forall\, A^j \subseteq \Gamma^j_C,\ meas(A^j) = 0 : \qquad meas({X^{(j)}}^{-1}(A^j)) = 0\,,$$

then the set of admissible displacements K *is closed.*

Proof. Carstensen, Scherf and Wriggers [15, Proposition 3.3] □

Remark 4.3. The only restriction for the parametrisations $X^{(j)}$ to get a suitable set of *admissible displacements*, i.e. K is closed, is given by Proposition 4.2. In Section 2.2 it was assumed that the parametrisations are smooth, i.e $X^{(j)} \in C^1(\Gamma_C, \Gamma^j_C)$. This isn't necessary for the definition of K, nevertheless in this thesis it's assumed that $X^{(j)}$ is smooth enough such that the operator

$$.^R : V \to (H^{\frac{1}{2}}(\Gamma_C))^d \qquad \forall\, j \in \{1,2\} \,:\, \overline{\Gamma^j_C} \cap \overline{\Gamma^j_D} = \emptyset$$

is continuous and surjective. If $\overline{\Gamma^j_C} \cap \overline{\Gamma^j_D} = \partial\Gamma^j_C)$, then $(H^{\frac{1}{2}}(\Gamma_C))^d$ is replaced by $(H^{\frac{1}{2}}_{00})^d$. Furthermore assume that the parametrisation is smooth enough such that $n \in C^{0,1}(\Gamma_C)^d$ and $g \in H^{\frac{1}{2}}(\Gamma_C)$.

If the parametrisations $X^{(j)} \in C^{0,1}(\Gamma_C, \Gamma^j_C)$ are Lipschitz continuous with Lipschitz continuous inverse and the Gram determinants J_j are bounded with bounded inverse (a.e.), then continuity and surjectivity of the operator $.^R$ is fulfilled, furthermore $g \in H^{\frac{1}{2}}(\Gamma_C)$ (see Eck [22, Voraussetzung 1.4]).

Remark 4.4. Assuming $n \in C^{0,1}(\Gamma_C)^d$, then every element $v \in H^{\frac{1}{2}}(\Gamma_C)^d$ can be decomposed into $v = v_N n + v_T$ with $v_N \in H^{\frac{1}{2}}(\Gamma_C)$ and $v_T \in H_T^{\frac{1}{2}}(\Gamma_C)$.

$$H_T^{\frac{1}{2}}(\Gamma_C) = \left\{ v \in H^{\frac{1}{2}}(\Gamma_C)^d \mid v_N = 0 \right\}$$

Due to the trace theorem (Theorem 3.5) this decomposition impose an isomorph from $H^{\frac{1}{2}}(\Gamma_C)^d$ into $H^{\frac{1}{2}}(\Gamma_C) \times H_T^{\frac{1}{2}}(\Gamma_C)$.

Remark 4.5. It's usually no restriction to assume that $\forall\, j \in \{1,2\} : \overline{\Gamma_C^j} \cap \overline{\Gamma_D^j} = \emptyset$, because Γ_C^j is an a-priori guess of the possible contact zone and can be chosen such that $\forall\, j \in \{1,2\}$: $\overline{\Gamma_N^j} \cap \overline{\Gamma_C^j} = \partial\Gamma_C^j$. With this assumption and the smoothness of the parametrisations $X^{(j)}$ it's possible to represent the set of *admissible displacements* via

$$K = \left\{ v \in V \mid g - v_N^R \in \mathcal{C}_+ \right\} ,$$

with $\mathcal{C}_+ := \left\{ q^* \in H^{\frac{1}{2}}(\Gamma_C) \mid q^* \geq 0 \quad \text{a.e.} \right\}$. The *cone* $\mathcal{C}_+$ was considered in Section 3.3. Because $\mathcal{C}_+$ is closed it's possible (Lemma 3.15) to represent it by $\mathcal{C}_+ = \mathcal{C}_+^{00}$. Thus the set of *admissible displacements* is nothing else than

$$K = \left\{ v \in V \mid \langle Bv - G, q \rangle_{Q' \times Q} \leq 0 \quad \forall\, q \in \mathcal{C}_+^0 \right\} . \tag{4.3}$$

Here Q' denotes the space $H^{\frac{1}{2}}(\Gamma_C)$ and B the linear, surjective and continuous operator $\cdot_N^R \in \mathcal{L}(V, Q')$. The polar cone the space Q'' is identified by Q, which is possible because Q is a reflexive *Banach space* ($\mathcal{C}_+^0 \in Q$). The function $g \in H^{\frac{1}{2}}(\Gamma_C)$ is denoted as $G \in Q'$ to point out its functional interpretation.

Definition 4.6. *$u \in K$ is called a weak solution of the contact problem iff*

$$\int_\Omega tr\,(\sigma(u)\epsilon(v-u))\,dx + \int_\Omega \langle\, \hat{F}, v-u \rangle_{l_2(\mathbb{R}^d)}\,dx + \int_{\Gamma_N} \langle\, L, v-u \rangle_{l_2(\mathbb{R}^d)}\,ds \geq 0 \quad \forall\, v \in K \tag{4.4}$$

holds for $\hat{F} \in H^{-1}(\Omega)$, $L \in H^{-\frac{1}{2}}(\Gamma_N)$

The following theorem proves that the *classical* formulation and the *weak* formulation are equivalent, assuming enough smoothness of the solution u.

Theorem 4.7. *Every solution of* (2.37) *is a weak solution. If a weak solution is sufficiently smooth, then it's a classical solution as well.*

Proof. 1. Let u be a solution of (2.37) and $v \in K$. Then by partial integration the following holds:

$$\int_\Omega \operatorname{tr}(\sigma(u)\operatorname{grad}(v-u))\,dx = \int_{\partial\Omega} \langle \sigma(u)^\nu,\, v-u \rangle_{l_2(\mathbb{R}^d)}\,ds - \int_\Omega \langle \operatorname{div}\sigma(u),\, v-u \rangle_{l_2(\mathbb{R}^d)}\,dx \tag{4.5}$$

Because σ is symmetric (2.13) it can easily be verified that

$$\int_\Omega \mathrm{tr}\,(\sigma(u)\,\epsilon(v-u))\,dx = \int_\Omega \mathrm{tr}\,(\sigma(u)\,\mathrm{grad}\,(v-u))\,dx$$

The surface integral in (4.5) can be simplified in the following way:

$$\begin{aligned} \int_{\partial\Omega} \langle \sigma^{(\nu)},\, v-u\rangle_{l_2(\mathbb{R}^d)}\,ds &= \int_{\Gamma_D\cup\Gamma_N\cup\Gamma_C^1\cup\Gamma_C^2} \langle \sigma^{(\nu)},\, v-u\rangle_{l_2(\mathbb{R}^d)}\,ds \\ &= \int_{\Gamma_N} \langle L,\, v-u\rangle_{l_2(\mathbb{R}^d)}\,ds + \int_{\Gamma_C^1\cup\Gamma_C^2} \langle \sigma^{(\nu)},\, v-u\rangle_{l_2(\mathbb{R}^d)}\,ds \end{aligned} \tag{4.6}$$

The remaining term is $\int_{\Gamma_C^j} \langle \sigma^{(\nu)},\, v-u\rangle_{l_2(\mathbb{R}^d)}\,ds$. This term is first transformed into its parameter domain Γ_C.

$$\begin{aligned} \int_{\Gamma_C^j} \langle \sigma^{(\nu^j)},\, v-u\rangle_{l_2(\mathbb{R}^d)}\,ds &= \int_{\Gamma_C} \langle \sigma^{(\nu^j)},\, v-u\rangle_{l_2(\mathbb{R}^d)} \circ X^{(j)}\,J_j\,dX \\ &= \int_{\Gamma_C} \langle \sigma,\, (v-u)\circ X^{(j)}\rangle_{l_2(\mathbb{R}^d)}\,dX \\ &= \int_{\Gamma_C} \langle \sigma_N+\sigma_T,\, (v-u)\circ X^{(j)}\rangle_{l_2(\mathbb{R}^d)}\,dX \end{aligned} \tag{4.7}$$

Using that $\sigma_T = 0$ and summing up gives that

$$\int_{\Gamma_C^1\cup\Gamma_C^2} \langle \sigma^{(\nu)},\, v-u\rangle_{l_2(\mathbb{R}^d)}\,ds = \int_{\Gamma_C} \sigma_N(v_N^R-u_N^R)\,dX = \int_{\Gamma_C} \sigma_N(v_N^R-g)\,dX \geq 0 \tag{4.8}$$

The assertion is proved by connecting (4.5), (4.6) and (4.8).

2. Now assume that $u\in K$ is a sufficiently smooth weak solution of (4.4). Let $\varphi \in C^\infty_{\partial\Omega}(\Omega)$ be arbitrary, $v = u\pm\varphi \;\Rightarrow\; v\in K$. Then from (4.19) follows:

$$\begin{aligned} 0 &\leq \int_\Omega \mathrm{tr}\,(\sigma(u)\,\epsilon(v-u))\,dx + \int_\Omega \langle \hat{F},\, v-u\rangle_{l_2(\mathbb{R}^d)}\,dx + \int_{\Gamma_N}\langle L,\, v-u\rangle_{l_2(\mathbb{R}^d)}\,ds \\ &= \pm\int_\Omega \mathrm{tr}\,(\sigma(u)\,\epsilon(\varphi)) + \langle \hat{F},\,\varphi\rangle_{l_2(\mathbb{R}^d)}\,dx \\ &\overset{(4.5)}{=} \pm\int_\Omega \langle\,\mathrm{div}\,\sigma(u)+\hat{F},\,\varphi\rangle_{l_2(\mathbb{R}^d)}\,dx\,, \end{aligned}$$

which implies

$$-\mathrm{div}\,\sigma(u) = \hat{F} \qquad \text{in } \Omega\,. \tag{4.9}$$

Now suppose that $\varphi\in C^\infty_{\partial\Omega/\Gamma_N}(\Omega)$, $v = u\pm\varphi \;\Rightarrow\; v\in K$. It follows from (4.4) and (4.9) that

$$0 \leq \pm\int_{\Gamma_N} \langle \sigma^{(\nu)} - L,\,\varphi\rangle_{l_2(\mathbb{R}^d)}\,ds \quad\Rightarrow\quad \sigma^{(\nu)} = L \quad \text{on } \Gamma_N\,. \tag{4.10}$$

The remaining boundary term which isn't fixed is

$$\int_{\Gamma_C^1 \cup \Gamma_C^2} \langle \sigma^{(\nu)},\, v - u \rangle_{l_2(\mathbb{R}^d)}\, ds \geq 0\,. \tag{4.11}$$

From (4.7) it's known that

$$\begin{aligned} 0 \;&\leq\; \int_{\Gamma_C^1 \cup \Gamma_C^2} \langle \sigma^{(\nu)},\, v - u \rangle_{l_2(\mathbb{R}^2)}\, ds \;=\; \int_{\Gamma_C} \sigma_N (v_N^R - u_N^R) + \langle \sigma_T,\, v_T^R - u_T^R \rangle_{l_2(\mathbb{R}^d)}\, dX \\ &=\; \int_{\Gamma_C} \langle \sigma^{(\nu^1)},\, v - u \rangle_{l_2(\mathbb{R}^d)} \circ X^{(1)}\, J_1 + \langle \sigma^{(\nu^2)},\, v - u \rangle_{l_2(\mathbb{R}^d)} \circ X^{(2)}\, J_2\, dX \end{aligned} \tag{4.12}$$

Because of the trace theorem (Remark 4.4) and the surjectivity of $.^R$ (see Remark 4.3) the following is valid:

$$\begin{array}{lll} i.) & \forall\, \phi \in H^{\frac{1}{2}}(\Gamma_C)^d\, \exists\, \varphi \in V\; : & \varphi \circ X^{(1)} = \varphi \circ X^{(2)} = \phi \\ ii.) & \forall\, \phi \in H_T^{\frac{1}{2}}(\Gamma_C)\;\, \exists\, \varphi \in V\; : & \varphi^R = \phi \quad (\wedge \quad \varphi_N^R = 0) \\ iii.) & \forall\, \phi \in H^{\frac{1}{2}}(\Gamma_C)\;\, \exists\, \varphi \in V\; : & \varphi_T^R = 0 \quad \wedge \quad \varphi_N^R = \phi \end{array}$$

Setting $v = u \pm \varphi$ and considering case $i.) - iii.)$ and using (4.12) follows

$i.)$ $\varphi^R = 0 \implies v \in K$

$$0 \;\leq\; \pm \int_{\Gamma_C} \langle\, J_1 \sigma^{(\nu^1)} \circ X^{(1)} - J_2 \sigma^{(\nu^2)} \circ X^{(2)},\, \phi \rangle_{l_2(\mathbb{R}^d)}\, dX\,. \tag{4.13}$$

$ii.)$ $\varphi_N = 0 \implies v \in K$

$$0 \;\leq\; \pm \int_{\Gamma_C} \langle \sigma_T,\, \varphi_T^R \rangle_{l_2(\mathbb{R}^d)}\, dX \quad \implies \quad \sigma_T = 0 \quad \text{on}\, \Gamma_C\,. \tag{4.14}$$

$iii.)$ With φ such that $\varphi_N^R \in \mathcal{C}_+$ follows

$$0 \;\leq\; \int_{\Gamma_C} \sigma_N \varphi_N^R\, dX = \langle \varphi_N^R,\, \sigma_N \rangle_{Q' \times Q} \quad \implies \quad \sigma_N \in \mathcal{C}_+^0\,. \tag{4.15}$$

From the assumed smoothness of the solution u it's deduced $\sigma_N \in \mathcal{C}_+^0 \cap L_2(\Gamma_C)$ and thus $\sigma_N \geq 0$ a.e. on Γ_C.

The last equation which have to be verified is the compatibility condition. This is done by setting φ such that $\varphi_N^R = g$ and $\varphi_T^R = 0$. With the setting $v = 2u - \varphi \in K$ everything is proved because

$$0 \;\leq\; \pm \int_{\Gamma_C} \sigma_N (u_N^R - \varphi_N^R)\, dX = \langle Bu - G,\, \sigma_N \rangle_{Q' \times Q} \;\leq 0\,. \tag{4.16}$$

Note that the partial ordering induced by the *cone* $\mathcal{C}_+$ is indeed a ordering.

A prove for contact with friction can be found in ECK [22, Satz 1.6] and an exact formulation of the necessary smoothness of u is presented in BOIERI, GASTALDI AND KINDERLEHRER [7, Theorem 2.2]. □

Using the following notation it's possible to write the *weak* formulation in a more abstract way

$$\begin{aligned} \langle Au, u - v\rangle_{V'\times V} &:= \int_\Omega \operatorname{tr}(\sigma(u)\epsilon(v-u))\,dx \\ \langle F, v - u\rangle_{V'\times V} &:= \int_\Omega \langle \hat{F}, v-u\rangle_{l_2(\mathbb{R}^d)}\,dx + \int_{\Gamma_N} \langle L, v-u\rangle_{l_2(\mathbb{R}^d)}\,ds\,. \end{aligned} \tag{4.17}$$

In this notation $A : V \to V'$ is an operator from V into its dual space V'. A is linear if σ is linear in u. The operator $F : V \to \mathbb{R} \in V'$ is linear and bounded and thus an element of the dual space V'. In the abstract notation the *weak* formulation *weak* formulation reads as follows:
Find $u \in K$ such that

$$\langle Au, v - u\rangle_{V'\times V} \geq \langle F, v - u\rangle_{V'\times V} \qquad \forall\, v \in K. \tag{4.18}$$

Remark 4.8. If $\langle Au, v\rangle_{V'\times V} = \langle Av, u\rangle_{V'\times V}$ then (4.18) is equivalent to the constraint minimisation problem:

$$u = \operatorname*{argmin}_{v\in K} J(v) \qquad \text{with} \qquad J(v) := \frac{1}{2}\langle Av, v\rangle_{V'\times V} - \langle F, v\rangle_{V'\times V} \tag{4.19}$$

Using (2.13) it's very easy to verify that A is symmetric. Because of this equivalence the *weak* formulation is often denoted as *primal* formulation

4.2 Some Abstract Results for Variational Inequalities

For considering variational inequalities in an abstract setting, an abstract elliptic inequality of the first and second kind is defined. For this abstract elliptic inequality some results concerning the existence and the uniqueness of its solution are presented. Then alternative variational formulations of elliptic inequalities and their mutual relations are discussed.

In the following, V will denote a real Hilbert space, V' its dual with the duality pairing $\langle .,.\rangle_{V'\times V}$. The norm on V will be denoted by $\|.\|_V$ and the dual norm by $\|.\|_{V'}$. Let $a : V \times V \to \mathbb{R}$ be a bilinear form and define the operator $A : V \to V'$ by $\langle Au, v\rangle_{V'\times V} = a(u, v)$. Let K be some *nonempty, closed and convex subset of* V.

Definition 4.9. *A triplet* $\{K, A, F\}$, $F \in V'$, *is called an* abstract elliptic variational inequality of the first kind. *A function* $u \in K$ *is called* solution *of* $\{K, A, F\}$ *iff*

$$\langle Au, v - u\rangle_{V'\times V} \geq \langle F, v - u\rangle_{V'\times V} \qquad \forall\, v \in K \tag{4.20}$$

In order to prove existence and uniqueness of the solution of $\{K, A, F\}$, assumptions on the linear operator A have to be added.
Suppose that the linear operator A is *bounded* on V:

$$\exists\, \alpha_2 > 0 : \; |\langle Au, v\rangle_{V'\times V}| \;\leq\; \alpha_2 \|u\|_V \,\|v\|_V \quad \forall\, u,\, v \in V \tag{4.21}$$

and *V-elliptic* on K:

$$\exists\, \alpha_1 > 0 : \; \langle Av, v\rangle_{V'\times V} \;\geq\; \alpha_1 \|v\|_V^2 \qquad \forall\, v \in K \tag{4.22}$$

Theorem 4.10. *Let K be a nonempty, closed, convex subset of a Hilbert space V. Then for each $v \in V$ there exists a unique element $u := Pv \in K$, named the projection of v onto K, such that the following equivalent assertions are valid:*

$$\|u - v\|_V \;\leq\; \|w - v\|_V \qquad \forall\, w \in K \tag{4.23}$$

respectively

$$\langle u - v, w - v\rangle_{V\times V} \;\geq\; 0 \qquad \forall\, w \in K \tag{4.24}$$

Furthermore $\forall\, w_1, w_2 \in K$ *the following holds:*

$$\|Pw_1 - Pw_2\|_V \;\leq\; \|w_1 - w_2\|_V \tag{4.25}$$

Proof. Kinderlehrer and Stampacchia [35] □

Using Theorem 4.10 and a similar proof as for the Lax Milgram Lemma, the Theorem of *Lions and Stampacchia* can be proved easily. In the Theorem of *Lions and Stampacchia*, there is no demand for symmetry of the appearing operator. Furthermore nonlinear problems can be dealt with. In Haslinger and Hlaváček and Nečas [29] an extension of this theorem on *reflexive Banach spaces* can be found (without proof).

Theorem 4.11 (Lions, Stampacchia). *Let V be a Hilbert space, $K \subset V$ a closed, convex, nonempty subset and $A : V \to V'$ a Lipschitz continuous and coercive (not necessarily linear) operator, i.e.* $\exists \alpha_1, \alpha_2 > 0$:

$$\begin{array}{lcll} \|Au - Av\|_{V'} & \leq & \alpha_2 \|u - v\|_V & \forall\, u, v \in K \\ \langle Au - Av, u - v\rangle_{V'\times V} & \geq & \alpha_1 \|u - v\|_V^2 & \forall\, u, v \in K\,. \end{array} \tag{4.26}$$

Then for each $F \in V'$ there exists a unique solution $u \in K$ of the variational inequality

$$\langle Au - F, v - u\rangle_{V'\times V} \;\geq\; 0 \qquad \forall\, v \in K \tag{4.27}$$

Furthermore the nonlinear solution operator is Lipschitz continuous with constant $\frac{1}{\alpha_1}$, i.e.

$$\|u_1 - u_2\|_V \;\leq\; \frac{1}{\alpha_1} \|F_1 - F_2\|_{V'}. \tag{4.28}$$

Proof. Kinderlehrer and Stampacchia [35] □

If A symmetric , i.e. $\langle Au, v\rangle_{V'\times V} = \langle Av, u\rangle_{V'\times V}$, then the problem $\{K, a, f\}$ is equivalent to the following one:

$$u = \operatorname*{argmin}_{v\in K} \mathcal{J}(v) \tag{4.29}$$

where $\mathcal{J} : V \to \mathbb{R}$ is the quadratic functional (*Ritz functional*)

$$\mathcal{J}(v) = \frac{1}{2}\langle Av, v\rangle_{V'\times V} - \langle F, v\rangle_{V'\times V}, \tag{4.30}$$

i.e. $u \in K$ solves $\{K, A, F\}$ iff it minimises $\mathcal{J}(v)$ over K.

Remark 4.12. Denote by $I_K : V \to \{0, +\infty\}$ the indicator function of K, i.e.:

$$I_K(V) = \begin{cases} 0 & \text{if } v \in K \\ +\infty & \text{elsewhere.} \end{cases}$$

Then (4.20) is formally equivalent to

$$\text{find } u \in V : \ \langle Au, v - u\rangle_{V'\times V} + I_K(v) - I_K(u) \geq \langle F, v - u\rangle_{V'\times V} \qquad \forall\, v \in V. \tag{4.31}$$

This motivates the following definition.

Definition 4.13. *Let $j : V \to \mathbb{R} \cup \{-\infty, \infty\}$ be a convex, lower semi continuous (i.e. $\forall\, u_n \to u \in V : \ j(u) \leq \liminf_{n\to\infty} j(u_n)$) and proper (i.e. $j(v) \neq -\infty$) functional on V. The quadruple $\{V, A, j, F\}$ is said to be an* abstract elliptic inequality of the second kind*. The function $u \in V$ is said to be a solution of $\{V, A, j, F\}$ iff*

$$\langle Au, v - u\rangle_{V'\times V} + j(v) - j(u) \geq \langle F, v - u\rangle_{V'\times V} \qquad \forall\, v \in V\,. \tag{4.32}$$

If $j = I_K$ is the indicator function of K, $\{V, A, j, F\}$ reduces to $\{K, A, F\}$. If moreover A is symmetric on V, the inequality $\{V, A, j, F\}$ is equivalent to the minimisation problem

$$u = \operatorname*{argmin}_{v\in V} \mathcal{J}(v)\,,$$

with $\mathcal{J} : V \to \mathbb{R} \cup \{-\infty, \infty\}$

$$\mathcal{J}(v) = \frac{1}{2}\langle Av, v\rangle_{V'\times V} + j(v) - \langle F, v\rangle_{V'\times V}\,. \tag{4.33}$$

There are a lot of results about existence and uniqueness of *abstract elliptic inequalities of the second kind*. This results are not presented in this thesis because the *abstract elliptic inequality of second kind* is only a subtotal.

Let Q be a *Banach* space, $G \in Q'$ and $B \in \mathcal{L}(V, Q')$, where V is defined as above and let $\geq$ be an order property on Q' with *positive cone* $\mathcal{C}_+$ which is closed and $\mathcal{C}_+^0$ be the corresponding *positive polar cone* in the dual space Q (Q'' is identified with Q). Let K be defined by

$$K = \{v \in V \mid Bv \leq G \text{ in } Q'\} . \tag{4.34}$$

Because $B \in \mathcal{L}(V, Q')$ it's obvious that K is convex and closed. If K is nonempty, property (4.21) is valid and if A is V-elliptic (4.22) then Theorem 4.11 is valid and thus $\{K, A, F\}$ is uniquely solvable. If $\mathrm{Rg}B = Q'$ it's obvious that K is nonempty.

It's nice to have existence and uniqueness, but it's difficult to calculate the solution, because K is only known formally, but an explicite characterisation is missing. Thus K has to be characterised. Consider the *abstract elliptic inequality of second kind* $\{V, A, I_K, F\}$ (which is equivalent to $\{K, A, F\}$) and assume that the *positive polar cone* $\mathcal{C}_+^0$ is known explicitely. From this the original problem can be reformulated as a *Saddle point problem.* This is done by writing

$$I_K = \sup_{p \in \mathcal{C}_+^0} \langle Bv - G, q \rangle_{Q' \times Q} . \tag{4.35}$$

The *abstract inequality of second kind* $\{V, A, I_K, F\}$ is now given by solving: find $(u, p) \in V \times \mathcal{C}_+^0$ such that:

$$L(u, p) = \inf_{v \in V} \sup_{q \in \mathcal{C}_+^0} \underbrace{\frac{1}{2} \langle Av, v \rangle_{V' \times V} + \langle Bv - G, q \rangle_{Q' \times Q} - \langle F, v \rangle_{V' \times V}}_{:= L(v,q)} . \tag{4.36}$$

Remark 4.14. This formulation has the advantage that the convex set K doesn't occur explicitly. Furthermore the minima is searched in a linear subspace. The price for this is that the problem size increases. Now both have to be solved, $u \in V$, which is the interesting variable, and the Lagrange multiplier $p \in \mathcal{C}_+^0$, which may not be from further interest.

4.3 Some Abstract Results for Saddle Point Problems

Let $K \subset V$, $N \subset Q$ be nonempty closed convex subsets and let $L : K \times N \to \mathbb{R}$ be a real functional defined on $K \times N$.

Definition 4.15. *The pair $(u, p) \in K \times N$ is called a* saddle point *of L iff*

$$L(u, q) \leq L(u, p) \leq L(v, p) \qquad \forall\, v \in K\, \forall\, q \in N .$$

Theorem 4.16. *A pair $(u, p) \in K \times N$ is a saddle point of L on $K \times N$ iff*

i. primal: $\quad \varphi(u) = \min_{v \in K} \varphi(v) \qquad \varphi(v) := \sup_{q \in N} L(v, q)$

ii. dual: $\quad \psi(p) = \max_{q \in N} \psi(q) \qquad \psi(q) := \inf_{v \in K} L(v, q)$

iii. $$L(u,p) = \varphi(u) = \psi(p)$$

Proof. see Ekland and Temam [23, IV Proposition 1.2] □

Proposition 4.17. *The set of saddle points of L is of the form $\mathcal{U} \times \mathcal{P}$ where $\mathcal{U} \subseteq K$ and $\mathcal{P} \subseteq N$.*

Proof. see Ekland and Temam [23, IV Propostion 1.4] □

In order to guarantee the existence, and eventually the uniqueness, of a saddle point, supplementary conditions are needed.

i. $\forall\, q \in N,\ v \mapsto L(v,q)$ is *convex and weakly lower semi continuous.*

ii. $\forall\, v \in K,\ q \mapsto L(v,q)$ is *concave and weakly upper semi continuous.*

Theorem 4.18. *Let the assumptions (i)-(ii) be satisfied and assume moreover,*

$$\exists\, q_0 \in N : \lim_{\|v\|_V \to \infty} L(v, q_0) = +\infty \tag{4.37}$$

$$\exists\, v_0 \in K : \lim_{\|q\|_Q \to \infty} L(v_0, q) = -\infty. \tag{4.38}$$

Then there exists a saddle point $(u,p) \in K \times N$ of L.

Proof. see Ekland and Temam [23, IV Proposition 2.2] □

Theorem 4.19. *The conclusions of Theorem 4.18 also hold if condition* (4.38) *is replaced by the condition*

$$\lim_{\|q\|_Q \to \infty} \inf_{v \in K} L(v,q) \to -\infty \tag{4.39}$$

Proof. see Ekland and Temam [23, IV Proposition 2.4] □

Theorem 4.20. *Let L be Gâteaux differentiable on $K \times N$ and let $\Phi : K \times N \to \mathbb{R}$ and L fulfils (i)-(ii). Then $(u,p) \in K \times N$ is a saddle point of $L + \Phi$ iff*

$$\begin{aligned} \left\langle \frac{\partial L}{\partial v}(u,p), v - u \right\rangle_{V' \times V} + \Phi(v,p) - \Phi(u,p) &\geq 0 \quad \forall\, v \in K \\ \left\langle \frac{\partial L}{\partial q}(u,p), q - p \right\rangle_{Q' \times Q} + \Phi(u,q) - \Phi(u,p) &\leq 0 \quad \forall\, q \in N \end{aligned} \tag{4.40}$$

Proof. see Ekland and Temam [23, IV Proposition 1.7] □

Back to the *abstract inequality of first kind* (4.19), where K is given by (4.34). $L(v,q)$ is then given by (4.36). This functional is G-differentiable on $V \times N$, where $N = \mathcal{C}^0_+$. Theorem 4.20 indicates that a saddle point $(u,p) \in V \times N$ of (4.36) can by characterised by the system

$$\begin{array}{lcll} \langle Au, v\rangle_{V'\times V} + \langle Bv, p\rangle_{Q'\times Q} & = & \langle F, v\rangle_{V'\times V} & \forall\, v \in V \\ \langle Bu - G, q - p\rangle_{Q'\times Q} & \leq & 0 & \forall\, q \in N. \end{array} \tag{4.41}$$

The question arises whether the *mixed* system (4.41) is uniquely solvable or not. The answer can't be given in general, thus two for this thesis important cases are distinguished. The first case is that $N \neq V$, then the existence and uniqueness is guaranteed if $\operatorname{Rg} B = Q'$ and A is V-elliptic. The second case is that $N = Q'$ and thus the inequality in (4.41) becomes an equality. In this case the existence and uniqueness is guaranteed if $\operatorname{Rg} B = Q'$ and A is elliptic on $\ker B$. Furthermore the unique solution of the equality case is bounded by $\|F\|_{V'}$, $\|G\|_{Q'}$. This two results are the topic of the next theorems. First an characterisation of $\operatorname{Rg} B = Q'$ is given in the case that V, Q are *Hilbert spaces*, which is the case in all examples we are interested in.

Proposition 4.21. *Let V, Q be two* Hilbert spaces *and $B \in \mathcal{V}, \mathcal{Q}'$, then the following statements are equivalent*

i. $Rg\, B \;=\; (ker\, B^*)^0$

ii. $Rg\, B^* \;=\; (ker\, B)^0$

iii. $\exists\, \beta \,>\, 0\, \forall\, v \in V \;:\quad \sup\limits_{q\in Q} \dfrac{\langle Bv, q\rangle_{Q'\times Q}}{\|q\|_Q} \;\geq\; \beta \|v\|_{V/ker B}$

iv. $\exists\, \beta \,>\, 0\, \forall\, q \in Q :\quad \sup\limits_{v\in V} \dfrac{\langle B^* q, v\rangle_{V'\times V}}{\|v\|_V} \;\geq\; \beta \|q\|_{Q/ker B^*}$

Proof. see Brezzi and Fortin [12, Proposition 1.2] □

For the inequality case, the next lemma proves all, what is needed to proof existence and uniqueness of the saddle point.

Lemma 4.22. *Suppose that* (4.21) *and* (4.22) (*with* $K = V$) *and*

$$\exists\, \beta \,>\, 0 : \quad \inf_{q\in Q} \sup_{v\in V} \frac{\langle Bv, q\rangle_{Q'\times Q}}{\|q\|_Q \|v\|_V} \geq \beta \tag{4.42}$$

hold. Then

$$\lim_{\|q\|_Q\to\infty} \inf_{v\in V} L(v,q) \to -\infty$$

Proof. see Kikuchi and Oden [34, Lemma 3.2] □

Remark 4.23. Condition (4.42) is due to the closed range theorem (Heuser [31, Satz 39.4]) equivalent to $\text{Rg}(B) = Q$.

Proposition 4.24. *Let $B \in \mathcal{L}(V, Q')$, $A \in \mathcal{L}(V, V')$. Assume that B fulfils the LBB condition (4.42) and assume that A is V-elliptic. Then there exists a unique solution $(u, p) \in V \times N$.*

Proof. Due to Lemma 4.22 condition (4.39) is valid and due to the assumption that A is V-elliptic also (4.37) is valid. Thus Theorem 4.19 guarantees the existence of a saddle point. With Theorem 4.20 this saddle point is also a solution of (4.41). The uniqueness is a consequence of the strict convexity of $v \to L(v, q)$ and the LBB condition (4.42). □

The equality case has a special importance in Chapter 6 and is thus also denoted.

Theorem 4.25. *Let V, Q be* Hilbert spaces*, $B \in \mathcal{L}(V, Q')$, $A \in \mathcal{L}(V, V')$ and $G \in Rg\,B$. Assume that $N = Q$ in (4.41), that $Rg\,B$ is closed in Q' and that A is elliptic on $ker\,B$. Then there exists a solution $(u, p) \in V \times Q$ of (4.41). Moreover (u, p) is bounded by*

$$
\begin{aligned}
\|u\|_V &\leq \frac{1}{\alpha_1}\|F\|_{V'} + \frac{1}{\beta}\left(1 - \frac{\alpha_2}{\alpha_1}\right)\|G\|_{Q'} \\
\|p\|_{Q/ker\,B^*} &\leq \frac{1}{\beta}\left(1 + \frac{\alpha_2}{\alpha_1}\right)\|F\|_{V'} + \frac{\alpha_2}{\beta^2}\left(1 + \frac{\alpha_2}{\alpha_1}\right)\|G\|_{Q'}\,.
\end{aligned}
$$

Proof. see Brezzi and Fortin [12, Proposition 1.3] □

Dual Formulation

In Theorem 4.16 the *dual* formulation was noted. For the body-body contact problem (L given by (4.36)) and with the assumption that A is V-elliptic on V, the *dual* formulation can be calculated explicitly. For this

$$
\psi(q) = \inf_{v \in V} \frac{1}{2}\langle Av, v\rangle_{V' \times V} + \langle Bv - G, q\rangle_{Q' \times Q} - \langle F, v\rangle_{V' \times V} \tag{4.43}
$$

has to be calculated. Because A was assumed to be V-elliptic (and bounded) Theorem 4.11 is valid and thus this equation is uniquely solvable for every q, B and F. The uniquely defined solution operator is denoted by A^{-1}. Because of Theorem 4.11 this operator is bounded and linear from $V' \to V$ ($A^{-1} \in \mathcal{L}(V', V)$). From the symmetry of A, the symmetry of A^{-1} is deduced. Substituting this result into (4.43) it follows

$$
\begin{aligned}
\psi(q) &= -\tfrac{1}{2}\left\langle B^*q + F, A^{-1}(B^*q + F)\right\rangle_{V' \times V} + \langle G, q\rangle_{Q' \times Q} \\
&= -\tfrac{1}{2}\left(\left\langle BA^{-1}B^*q, q\right\rangle_{Q' \times Q} + 2\left\langle BA^{-1}F, q\right\rangle_{Q' \times Q} + \left\langle F, A^{-1}F\right\rangle_{V' \times V}\right) + \langle G, q\rangle_{Q' \times Q} \\
&= -\tfrac{1}{2}\left\langle BA^{-1}B^*q, q\right\rangle_{Q' \times Q} - \left\langle BA^{-1}F - G, q\right\rangle_{Q' \times Q} - \tfrac{1}{2}\left\langle F, A^{-1}F\right\rangle_{V' \times V}
\end{aligned}
$$

By changing the sign and sup $\to$ inf, the *dual* formulation looks like

$$p = \operatorname*{argmin}_{q \in N} \psi(q)\,,$$

with

$$\psi(q) = \frac{1}{2}\left\langle BA^{-1}B^{*}q, q\right\rangle_{Q' \times Q} + \left\langle BA^{-1}F - G, q\right\rangle_{Q' \times Q} \tag{4.44}$$

If p is known, the solution u can be calculated by applying A^{-1} to $B^{*}p + F$. Note B^{*} is the adjoint operator of B with $B^{*} \in \mathcal{L}(Q, V')$. Kikuchi and Oden[34] denote the *dual* formulation as *reciprocal* formulation.

4.4 Existence and Uniqueness of the Body-Body Contact Problem

Theorem 4.11 guarantees existence and uniqueness if A, given by (4.17), is elliptic on K (Boundness of A is easy to verify). Also the ellipticity of A on K may be too much. There are body-body contact problems which are not elliptic on K having a unique solution. Ellipticity isn't necessary because there are more general results than Theorem 4.11. Some of them are presented in the references given at the beginning of this section. If A isn't K-elliptic, then the existence and uniqueness results are usually based on the coercitivity of the *Ritz functional* $\mathcal{J}(v)$ ((4.30). Especially for the body-body contact problem existence and uniqueness results are presented in Kikuchi and Oden [34], Haslinger, Hlaváček and Nečas [29], Boieri, Gastaldi and Kinderlehrer [7] and Eck [22].

For this thesis only examples are considered which have *Dirichlet boundaries* on both bodies, i.e. $|\Gamma_D^1|$, $|\Gamma_D^2| > 0$. In this case it's possible to prove that A is V-elliptic. This is done by considering the null space of A, which are, due to the next lemma, the *rigid* body motions.

Lemma 4.26. *Let $\Omega \in C^{0,1}$. Then the following conditions are equivalent:*

- $\forall\, v \in H^1(\Omega)\, \forall\, \in \{1, \dots d\}$: $\quad \epsilon_{ij}(v) = 0$ *in* $L_2(\Omega)$
- $\exists\, a, b \in \mathbb{R}^d$: $\quad v(x) = a + b \times x$ *a.e. in* Ω,

where $\times$ denotes the vector product and $x \in \Omega$ is the position vector of a point.

Proof. see Kikuchi and Oden [34, Lemma 6.1] □

The most important inequality in elastic mechanics, from which the most ellipticity results of A are deduced is the *Korn's inequality.*

Theorem 4.27. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with $\partial\Omega \in C^{0,1}$. Then there exists a positive constant c, independent of v, such that*

$$\int_\Omega tr(grad v\, grad v)\, dx \;\leq\; C \int_\Omega tr(\epsilon(v)\, \epsilon(v))\, dx + \|v\|^2_{0,\Omega} \tag{4.45}$$

for every $v \in W^{1,p}(\Omega)$, $p \in]1, \infty[$.

Proof. for a sketch of the proof see KIKUCHI AND ODEN [34, Theorem 5.13] □

Using *Korn's inequality* (Theorem 4.27) and the assumption $|\Gamma_1^1|, |\Gamma_D^2| > 0$, the V-ellipticity follows as a direct consequence of Theorem 3.18. Theorem 3.18 is used to prove that

$$\int_\Omega \mathrm{tr}(\epsilon(v)\, \epsilon(v))\, dx + \|v\|^2_{0,\Omega} \simeq \int_\Omega \mathrm{tr}(\epsilon(v)\, \epsilon(v))\, dx\,,$$

which follows directly by setting

$$|\,.\,|_A^2 \;=\; \int_\Omega \mathrm{tr}(\epsilon(v)\, \epsilon(v))\, dx\,,$$

$$|\,.\,|_B \;=\; \left| \int_{\Gamma_D} v\, ds \right|$$

and noting that X_0 is the set of *rigid body motions* given in Lemma 4.26.

Remark 4.28. Because A is V-elliptic and it was assumed that the parametrisations $X^{(j)}$ is smooth enough such that $\mathrm{Rg}\, B = Q'$, also the *mixed* formulation (4.41) has, due to Proposition 4.24 , a unique solution and thus the *dual* formulation as well. This fact will be essential for the following chapters.

Chapter 5

Some Results for Finite Element Analysis

To solve variational inequalities numerically, the infinite space V (Q) for the primal (dual) formulation and the space $V \times Q$ for the mixed formulation has to be reduced to a finite one. There are a lot of methods to do this, but the common one and the best suited method for numerical computation is the Finite Element Method (FEM). For details of FEM refer for the primal (dual) problem to CIARLET [16] and for the mixed problem to BREZZI AND FORTIN [12]. An important property of the FEM is that the convergence results of the discrete to the continuous solution usually depend only on the best approximation error. The best approximation error is easy to estimate for FEM due to the local property of FEM and well known interpolation operators. For variational inequalities some convergence results are presented in BREZZI, HAGER AND RAVIART [13],[14] and FALK [24]. For interpolation operators refer to SCOTT AND ZHANG [45], CLEMÉNT [18] and DUPONT AND SCOTT [21].

In Section 5.1 the finite element space is introduced and some notations are defined. To keep the writing short, the notation is important for the next chapters. An important property of the FEM is that there exists several interpolation operators. For the discretisation of the convex set K of the body-body contact problem, these interpolation operators are essential. Thus the most important one, at least for this thesis, the *nodal* interpolation operator and the interpolation operators of *Scott-Zhang* type are introduced in Section 5.2. To guarantee existence and uniqueness of the discretised solution, corresponding assumptions as in Section 4.3 are needed. In Section 5.3 an abstract condition (*Fortin's criterion*) is presented which guarantees the the discrete LBB condition holds uniformly in h. Finally it's important to know whether the discrete solution converges to the exact one or not. For this some results are presented in Section 5.4.

5.1 Finite Element Spaces

For computing, elements in *Sobolev spaces* have to be approximated by elements in finite dimensional function spaces. One possibility are finite element spaces (FE-spaces). This kind of finite dimensional spaces are very common and should be introduced in this section. For more details refer to CIARLET [16]. The FEM is a very general method to construct finite dimensional subspaces for *Sobolev spaces* and well suited for a lot of practical applications.

For simplicity and to avoid technical difficulties assume that $\Omega \subseteq \mathbb{R}^d$ is a polygonal shaped and bounded domain. The first characteristic of the FEM is that a triangulation τ_h is established over the set $\overline{\Omega}$, i.e. the set $\overline{\Omega}$ is subdivided into a finite number of subsets T, called finite elements, in such a way that the following properties are satisfied.

- $\overline{\Omega} = \bigcup_{T \in \tau_h} T$
- $\forall T \in \tau_h : \quad \mathrm{cl}\, T = T \wedge \mathrm{int}\, T \neq \emptyset$
- $\forall T, K \in \tau_h, T \neq K : \quad \mathrm{int}\, T \cap \mathrm{int}\, K = \emptyset$
- $\forall T \in \tau_h : \quad \partial T \in C^{0,1}$.

For this thesis and for simplicity it's enough to restrict the finite elements T to segments in $1D$, triangles in $2D$ and tetrahedra in $3D$.

For notation define the patches

$$\begin{aligned} \omega_T &= \bigcup_{\substack{K \in \tau_h \\ K \cap T \neq \emptyset}} T\,, \\ \omega(x) &= \bigcup_{\substack{K \in \tau_h \\ x \in T}} T\,, \end{aligned}$$

the set of edges $\mathcal{E}_h$ by

$$\mathcal{E}_h = \{ E \subset \Omega \,|\, \exists T, K \in \tau_h : E = T \cap K \vee E = T \cap \partial\Omega \} \tag{5.1}$$

and split the set of edges $\mathcal{E}_h$ into disjoint sets $\mathcal{E}_h^{\Omega}$, $\mathcal{E}_h^{D}$, ..., such that

$$\begin{aligned} \mathcal{E}_h^{\Omega} &= \{ E \in \mathcal{E}_h \,|\, \exists T, K \in \tau_h : E = T \cap K \}\,, \\ \mathcal{E}_h^{D} &= \{ E \in \mathcal{E}_h \,|\, \exists T \in \tau_h : E = T \cap \overline{\Gamma_D} \}\,. \end{aligned}$$

Remember that $\Gamma_D \ldots \in \partial\Omega$ is a disjoint splitting of the boundary. These definitions will simplify the writing in the following chapters.

For the FEM each element $T \in \tau_h$ is interpreted as the image of the mapping $x^T(\xi)$ from a reference element $T^{(R)}$. The usual case is that x^T is an affine linear mapping.

Definition 5.1.

- *A triangulation τ_h is called* conforming*, iff the intersection of two different elements is either empty, or contains one common vertex, one common edge or one common face.*
- *A* conforming *triangulation is* shape regular*, iff for all elements the condition number of the Jacobian is bounded, i.e.*

$$\left\| \frac{dx^T(\xi)}{d\xi} \right\|_{l_2(\mathbb{R}^{d\times d})} \left\| \left(\frac{dx^T(\xi)}{d\xi} \right)^{-1} \right\|_{l_2(\mathbb{R}^{d\times d})} \preceq 1 \qquad \forall\, T \in \tau_h \,\forall\, \xi \in T\,. \tag{5.2}$$

- *The local mesh size $h(x)$ is defined for $x = x^T(\xi)$ as*

$$h(x) = \left\| \frac{dx^T(\xi)}{d\xi} \right\|_{l_2(\mathbb{R}^{d\times d})}, \tag{5.3}$$

and

$$h_T = \sup_{x\in T} h(x)\,.$$

- *A triangulation τ_h is* quasi-uniform*, iff it's* shape regular *and there exists one global $h > 0$ such that*

$$h \preceq h(x) \preceq h \qquad \forall x \in \Omega\,. \tag{5.4}$$

Let X_h be any *finite-dimensional* space of functions defined over $\overline{\Omega}$. With such a finite element space X_h the (finite dimensional) spaces

$$P_T = \{\, v_h|_T \;\mid\; v_h \in X_h \,\}$$

are defined, where $T \in \tau_h$.

The second basic aspect of FEM is that the spaces P_T, $T \in \tau_h$ contain polynomials. For this define shape functions on the reference element. It's enough, for this thesis, to consider shape functions of full polynomial type. This shape functions are denoted by $\mathcal{P}^r$.

$$\mathcal{P}^r = \{\, p = \sum_{|\alpha|\le r} a_\alpha\, x_1^{\alpha_1} \ldots x_d^{\alpha_d} \quad a_\alpha \in \mathbb{R} \,\}\,. \tag{5.5}$$

The third basic aspect of the FEM is that there exists at least one "canonical" basis in the FE space V_h whose corresponding basis functions have supports which are as "small" as possible. It's being implicitly understood that these basis functions can be easily described.

Fix the polynomial degree r for $\mathcal{P}^r$ and consider the FE space V_h which is defined by

$$V_h = \{\, v_h \in C(\overline{\Omega}) \mid v_h|_T \in P^r \quad \forall\, T \in \tau_h \,\}\,. \tag{5.6}$$

Note that this finite dimensional space V_h is subset of $H^1(\Omega)$ (see CIARLET [16, Theorem 2.1.1]). It can be represented as the span of local ansatz functions $\{\varphi_i\}$ with corresponding set of nodes $\mathcal{N}_h$ such that for $x_i \in \mathcal{N}_h$

$$\varphi_i(x_j) = \delta_{ij} .$$

For linear *Lagrange* elements (i.e. $r = 1$), $\mathcal{N}_h$ can be chosen as the union of all corner points of the elements $T \in \tau_h$. The corresponding ansatz functions $\{\varphi_i\}$ are local and defined on the patch $\{\omega(x_i)\}$. Also the set of nodes $\mathcal{N}_h$ is split into disjoint parts $\mathcal{N}_h^\Omega$, $\mathcal{N}_h^D$, ..., similar as the set of edges $\mathcal{E}_h$. An important property of FE spaces for *shape regular* triangulation τ_h is the so called inverse inequality. I.e. it's possible to estimate higher order *Sobolev norms* on elements $T \in \tau_h$ by lower one.

Theorem 5.2. *Let τ_h be a* shape regular *mesh, V_h be a FE space. Then the following inequality is valid.*

$$\|v_h\|_{k,T} \preceq h_T^{l-k} \|v_h\|_{l,T} \qquad \forall\, 0 \le l \le k \le \tag{5.7}$$

Proof. see CIARLET [16, Theorem 3.2.6] □

5.2 Local Interpolation Operators

To approximate a function in a *Sobolev space* by some finite element functions a mapping $I_h : H^m(\Omega) \to V_h$ is needed. The mapping should be local, i.e. $(I_h u)|_T$ should only depend on $u|_{\tilde{T}}$, where $\tilde{T}$ is close to and not much larger than T. The approximation shell become better as the image norm gets weaker. The optimal approximation is

$$\|v - I_h v\|_{k,T} \preceq h_T^{j-k} \|v\|_{j,\tilde{T}} \qquad \forall\, v \in V\, \forall\, 0 \le k \le j \le m \le r+1 , \tag{5.8}$$

for proper integers k and j. r is the maximal polynomial degree of the ansatz functions, i.e. $\varphi_i \in \mathcal{P}^r(T)$. One further property, which isn't necessary but very common, is that the interpolation operator I_h is idempotent, i.e. $I_h^2 = I_h$.

Every linear interpolation operator I_h can be represented as

$$I_h v = \sum_{x_i \in \mathcal{N}_h} l_i(v)\varphi_i \qquad \forall\, v \in V , \tag{5.9}$$

where $l_i \in V'$ is some linear functional. Note that $l_i(v) = I_h(v)(x_i)$ and its norm is in general mesh dependent. For a local mapping the linear functional l_i has to be local, i.e. $\forall\, v \in V$, $v|_{\tilde{T}} = 0 :\quad l_i(v) = 0$ and for I_h idempotent $l_i(I_h v) = l_i(v) \quad \forall\, x_i \in \mathcal{N}_h$. The hard property for an interpolation operator I_h is equation (5.8). In the following two examples of interpolation operators are presented which fulfils (5.8).

- The classical interpolation operator is the nodal interpolation operator I_N. From Theorem 3.4 it's known that $H^m(\Omega)$ is continuously embedded into the continuous functions $C(\Omega)$, if $m > \frac{d}{2}$. Thus it's possible to evaluate every function $v \in H^m(\Omega)$ in every arbitrary point $x \in \Omega$ with $v(x) \leq \max_{y\in\Omega} |v(y)| \preceq \|v\|_{m,\Omega}$. I.e. the linear functional $l_x(v) := v(x)$ is element of the space $H_0^{-m}(\Omega)$. Especially $l_i(\,.\,) := l_{x_i}(\,.\,) \quad x_i \in \mathcal{N}_h$ is element of $H_0^{-m}(\Omega)$. With this it's possible to define the nodal interpolation operator I_N

$$I_N v := \sum_{x_i \in \mathcal{N}_h} v(x_i)\varphi_i \,. \tag{5.10}$$

  This operator is obviously a local interpolation operator and idempotent. In Ciarlet [16, Theorem 3.2.1] it's proved that this interpolation operator also fulfils (5.8). The main reason that (5.8) is valid for the nodal interpolation operator is that

$$(I - I_N)p = 0 \quad \forall\, p \in \mathcal{P}^r(T)\,\forall\, T \in \tau_h$$

  and that $l_i(v) \preceq \|v\|_{m,\Omega}$. Thus the *Bramble Hilbert* lemma (see Lemma 3.20) is valid and with the transformation of the element T to its reference element $T^{(R)}$ everything follows.

  The big disadvantage of the nodal interpolation operator is that m has to be greater than $\frac{d}{2}$, which is especially not fulfilled for 2D and 3D examples with $m = 1$.
  Note that nodal interpolation provides the same approximation result for the boundary as for the domain.

- An other possibility is the *Scott-Zhang* type interpolation operator. These operators are defined as follows. For each node $x_i \in \mathcal{N}_h$ define a set σ_i, with $x_i \in \sigma_i$. It can be a subset of non-zero measure, but also a manifold. Define the $L_2(\sigma_i)$-orthogonal projection Π_i^k onto $\mathcal{P}^k(\sigma_i)$. Then define the linear functionals $l_i(v) := \Pi_i^k(v)(x_i)$ and thus the interpolation operator becomes

$$I_{SZ} v := \sum_{x_i \in \mathcal{N}_h} \Pi_i^k(v)(x_i)\varphi_i \,. \tag{5.11}$$

  This operator is idempotent if $V_h|_{\sigma_i} \subseteq \mathcal{P}^k(\sigma_i)$ and then also local. Two examples to define the set σ_i for the node $x_i \in \mathcal{N}_h$ are given in Figure 5.1. The *Scott-Zhang* projection is well defined for the *Sobolev space* $L_2(\Omega)$ iff all sets σ_i have non-zero measure in $\mathbb{R}^d$, in the other case it's well defined on $H^m(\Omega)$ with $m > \frac{1}{2}$, due to the trace theorem (see Theorem 3.5). If additionally

$$|\sigma_i| = \begin{cases} O(h_T^d) & \dim\sigma_i = d \\ O(h_T^{d-1}) & \dim\sigma_i = d-1 \end{cases},$$

  then the approximation inequality (5.8) holds for $1 \leq m \leq \max\{r+1, k+1\}$. For proof refer to Scott and Zhang [45].

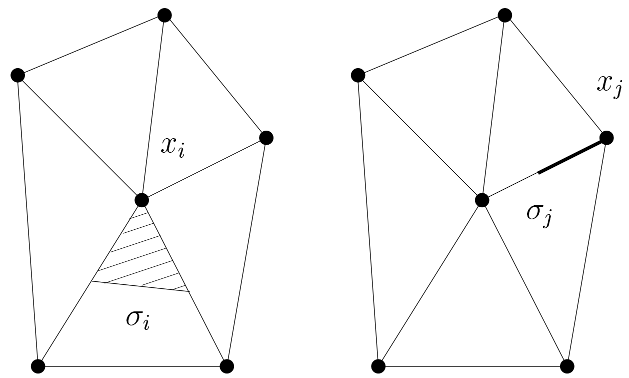


Figure 5.1: construction of a SCOTT-ZHANG operator

Note that if $\sigma_i \subset \partial\Omega \quad \forall\, x_i \in \mathcal{N}_h \cap \partial\Omega$, then the *Scott-Zhang* interpolation operator is also one at the boundary $\partial\Omega$ with the same properties as in Ω.

Remark 5.3. Due to the fact that Hilbert type *Sobolev spaces* of fractional order can be represented as interpolation spaces of two *Sobolev spaces* with an integer order (see Theorem 3.13) and due to Theorem 3.11, the approximation result (5.8) can be extended to sobolev indices with fractional order. This is often denoted as *scaling argument.*

5.3 Existence and Uniqueness of the Discretised Variational Inequality

For the $\{K, A, F\}$ denoted as saddle point formulation (4.41), with

$$K = \{v \in V \mid \langle Bv - G, q\rangle_{Q' \times Q} \leq 0 \quad \forall, q \in N\},$$

the existence and uniqueness was guaranteed, due to Proposition 4.24, if A is V-elliptic and B fulfils the LBB condition (4.42). For the equality case, i.e. $N = Q$, the existence and uniqueness was guaranteed, due to Theorem 4.25, if A is $\operatorname{Ker} B$-elliptic and B fulfils the LBB condition (4.42). This results are also valid in the discretised case. Thus it's enough to prove a discrete LBB condition, i.e. $BV_h = Q_h'$, and for the equality case the discrete kernel ellipticity. To achieve convergence results for the discretised system it's necessary that the discrete LBB condition and the discrete kernel ellipticity holds uniformly, i.e. independent of the mesh parameter h. The following theorem guarantees an uniform discrete LBB condition in a lot of applications.

Theorem 5.4 (Fortin's criterion). *Let V, Q be a stable pair, i.e. the LBB condition*

$$\sup_{v \in V} \frac{b(v,q)}{\|v\|_V} \succeq \|q\|_Q$$

is fulfilled and V_h, Q_h the corresponding FE spaces. Furthermore assume the existence of a family of uniformly continuous operators $I_h^F : V \to V_h$ satisfying

$$\begin{array}{rcl} b(I_h^F w - w, q_h) & = & \forall\, q_h \in Q_h\,, \\ \|I_h^F v\|_V & \preceq & \|v\|_V\,. \end{array} \tag{5.12}$$

Then the FE spaces V_h, Q_h are a uniform stable pair, i.e.

$$\sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_V} \succeq \|q_h\|_Q \tag{5.13}$$

Proof. see Brezzi Fortin [12, II Proposition 2.8] ☐

For the inequality case the last theorem is enough to guarantee existence and uniqueness. If

$$\ker_h B = \{\, v_h \in V_h \,|\, \rangle B v_h, q_h \rangle_{Q' \times Q} = 0 \qquad \forall\, q_h \in Q_h \,\}$$

is subspace of $\ker B$ then Theorem 5.4 is also enough for the equality case, to guarantee existence and uniqueness. But in general $\ker_h B \not\subseteq \ker B$ and thus the discrete kernel ellipticity has to be verified by hand.

5.4 Convergence Results

In the last sections the variational inequality was discretised and the existence of a unique solution was guaranteed. But the hole work will be useless if the discretised solution is no approximation of the continuous one. This section is concerned with some abstract results which guarantees that the discretised solution approximates the continuous one.

In the hole section e denotes the discretisation error in the *primal* variable, i.e.

$$e := u - u_h\,.$$

Most of the results and inequalities presented, are concerned with the convergence of the *primal* variable. Only for the equality case a convergence result for both, the *primal* and *dual* variable is presented. Some of the inequalities, which are presented are well suited to construct a-posteriori error estimators, at least for the equality case. The convergence result, presented in this thesis, for the inequality case requires a special representation of the discretised convex set K_h, but that's enough for the problems considered here.

$$K_h = \{v_h \in V_h \,|\, \langle B v_h - G, q_h \rangle_{Q' \times Q} \le 0 \qquad \forall\, q_h \in N_h\}\,. \tag{5.14}$$

Before presenting some inequalities of the discretisation error in the A energy norm, the notion of the *approximation of K* is defined.

Definition 5.5. *A family* $(K_h)_{h\in H}$*, where* $K_h \subset V_h$ *are non-empty, closed and convex subsets of* V_h*, is said to be an* approximation of K *iff*

- $\forall\, v \in K\, \exists\, v_h \in K_h : \lim_{h\to 0+} v_h = v \; in\, V$ (5.15a)
- *if* $(v_h)_{h\in H}$, $v_h \in K_h$ *is such that* $v_h \rightharpoonup v,\ h \to 0+\ \ in\, V \Rightarrow\ v \in K.$ (5.15b)

The next lemmas are the basis of proofing convergence results in the inequality case. Some of these lemmas are used later, to prove convergence results for the body-body contact problem with *nodal constraints* as discretisation.

Lemma 5.6. *Let* (u, p) *be solution of* (4.41) *and* (u_h, p_h) *be the corresponding one of the discretised system, with* K_h *given by* (5.14)*, then the following error estimate is valid for all* $v_h \in K_h$ *and* $q_h \in N_h$*.*

$$\begin{aligned}\langle Ae, e\rangle_{V'\times V} \;\; \le \;\; & \langle Ae, u - v_h\rangle_{V'\times V} + \langle B(u_h - v_h), p - q_h\rangle_{Q'\times Q} \\ & + \langle B(u - v_h), q_h\rangle_{Q'\times Q} + \langle Bu - G, p - q_h\rangle_{Q'\times Q}\end{aligned} \tag{5.16}$$

Proof. see Brezzi, Hager and Raviart [14, II Theorem 2.2] □

Lemma 5.7. *Let* (u, p) *be solution of* (4.41) *and* (u_h, p_h) *be the corresponding one of the discretised system. Then the residual* $e = u - u_h$ *in the energy norm has the following representations for all* $e_h \in V_h$, $q_h \in N_h$, $q \in N$

$$\begin{aligned}\langle Ae, e\rangle_{V'\times V} \;\; = \;\; & \{\langle F, e - e_h\rangle_{V'\times V} - \langle Au_h, e - e_h\rangle_{V'\times V} - \langle B(e - e_h), p_h\rangle_{Q'\times Q}\} \\ & + \{\langle Bu - G, p_h\rangle_{Q'\times Q}\} + \{\langle Bu_h - G, p\rangle_{Q'\times Q}\}\end{aligned} \tag{5.17a}$$

$$\begin{aligned}\langle Ae, e\rangle_{V'\times V} \;\; \le \;\; & \{\langle F, e - e_h\rangle_{V'\times V} - \langle Au_h, e - e_h\rangle_{V'\times V} - \langle B(e - e_h), p_h\rangle_{Q'\times Q}\} \\ & + \{\langle Bu - G, p_h - q\rangle_{Q'\times Q}\} + \{\langle Bu_h - G, p - q_h\rangle_{Q'\times Q}\}\end{aligned} \tag{5.17b}$$

Proof. From (4.41) it follows that

$$\langle Au - u_h, v_h\rangle_{V'\times V} \; = \; \langle Bv_h, p_h - p\rangle_{Q'\times Q} \qquad \forall\, v_h \in V_h\,.$$

Thus the residual $(e := u - u_h)$ in the energy norm can be estimated by

$$\begin{aligned}\langle Ae, e\rangle_{V'\times V} \;\; = \;\; & \langle Au, e - e_h\rangle_{V'\times V} - \langle Au_h, e - e_h\rangle_{V'\times V} - \langle Be_h, p - p_h\rangle_{Q'\times Q} \\ = \;\; & \{\langle F, e - e_h\rangle_{V'\times V} - \langle Au_h, e - e_h\rangle_{V'\times V} - \langle B(e - e_h), p_h\rangle_{Q'\times Q}\} \\ & + \langle Bu, p_h - p\rangle_{Q'\times Q} + \langle Bu_h, p - p_h\rangle_{Q'\times Q} \\ = \;\; & \{\langle F, e - e_h\rangle_{V'\times V} - \langle Au_h, e - e_h\rangle_{V'\times V} - \langle B(e - e_h), p_h\rangle_{Q'\times Q}\} \\ & + \{\langle Bu, p_h\rangle_{Q'\times Q} - \langle G, p_h\rangle_{Q'\times Q}\} + \{\langle Bu_h, p\rangle_{Q'\times Q} - \langle G, p\rangle\}_{Q'\times Q}\end{aligned}$$

The inequality (5.17b) is a direct consequence of the last estimate and $\langle Bu - G, q\rangle_{Q'\times Q} \le 0$, $\langle Bu_h - G, q_h\rangle_{Q'\times Q} \le 0$. □

Lemma 5.8. *It holds that* $\forall\, v \in K\ \forall\, v_h \in K_h$

$$\begin{aligned} \langle Ae, e\rangle_{V'\times V} \;\leq\; & \langle F, u - v_h\rangle_{V'\times V} + \langle F, u_h - v\rangle_{V'\times V} \\ & +\langle Ae, u - v_h\rangle_{V'\times V} + \langle Au, v - u_h\rangle_{V'\times V} + \langle Au, v_h - u\rangle_{V'\times V}\,. \end{aligned} \tag{5.18}$$

Proof. see Haslinger and Hlaváček and Nečas [29, Lemma 4.1] □

Theorem 5.9. *Let* $u \in K$ *and* $u_h \in K_h$ *be solutions of* $\{K, A, F\}$ *and* $\{K_h, A, F\}$*, respectively. Let the family* $\{K_h\}$ *be* an approximation of K*, i.e.* (5.15a)-(5.15b) *is fulfilled. Then*

$$\lim_{h\to+0} \|u - u_h\|_V = 0\,.$$

Proof. see Haslinger and Hlaváček and Nečas [29, Theorem 4.1] □

Condition (5.15a) is some minimal assumption which should be fulfilled to guarantee a convergence result without assuming any properties for the solution u.

In the equality case (4.41) has nice convergence properties, which will be very important in the next chapter.

Theorem 5.10. *Assume* $N = Q$ *(unrestricted mixed problem), that* B *fulfils the LBB condition* (4.42) *and* A *should be* $\ker B$*-elliptic* (4.22)*. Furthermore assume the uniform discrete LBB condition* (5.13) *and also the uniform* $\ker_h B$*-ellipticity for* A*. Let* (u, p) *be the unique solution of* (4.41) *and* (u_h, p_h) *be the unique solution of the corresponding discretised system. Then the following convergence result is valid.*

$$\|u - u_h\|_V + \|p - p_h\|_Q \preceq \inf_{v_h\in V_h} \|u - v_h\|_V + \inf_{q_h\in Q_h} \|p - q_h\|_Q \tag{5.19}$$

Proof. see Brezzi and Fortin [12, II Propositon 2.6-2.7] □

One draw back of this section is that the convergence result presented for the variational inequality isn't valid for the body-body contact problem and the discretisation (*nodal constraints*) chosen in this thesis. For the body-body contact problem the inequalities presented for the *primal* variable are from greater interest.

Chapter 6

Intermezzo Non-Matching Grids

Standard discretisation of the body-body contact problem will usually result into a problem with non-matching grids at the contact boundary. There are at least two techniques to handle non-matching grids. The first is the *Mortar* method and the second one is *Nodal constraints* (on a master surface). The *Mortar* method is well developed for equality constraints and give optimal convergence results (see BELGACEM [4]). There are also some results for contact problems (see BELGACEM, HILD AND LABORDE [6] [5]). One disadvantage is that this method is hard to implement, especially the three dimensional case. *Nodal constraints* have the advantage that they are easy to implement but, to the knowledge of the author, there are no theoretical optimal convergence results available. This chapter follows SCHÖBERL et al [28] and presents the analysis of a simple example with *Nodal equality constraints*. The main result will be that *Nodal constraints* with a simple stabilising term has optimal convergence for *Lagrange* elements, in the primal variable, at least in 2 dimensions.

In Section 6.1 a *mixed* problem, equivalent to the *Laplace* problem (see RAVIART AND THOMAS [41]), is presented, which enables the handling of *non-matching grids*. Introducing an abstract interpolation operator and assuming its existence, it's possible to introduce *Nodal constraints* as a special choice of ansatz functions in the dual space. This and the validity of the standard requirements for discretised *mixed* problems is presented in Section 6.2. The convergence result which is achieved due to this discretisation isn't optimal in the primal variable. In Section 6.3 the *mixed* problem is consistently modified, such that all standard results, apart of the uniform V_{0h}- ellipticity, are trivially valid. Assuming this V_{0h}-ellipticity, optimal convergence results are achieved for *Nodal constraints* and linear *Lagrange* elements. In Section 6.4 both, the existence of the abstract interpolation operator and the assumed uniform V_{0h}-ellipticity of the consistently modified problem, in 2 dimensions are proved. In Section 6.5 it's explained how to implement the consistently modified problem efficient (at least for a uniform mesh) and an numerical example is presented which confirms the optimal convergence property of the consistently modified problem.

6.1 Non-matching Grid for a Laplace Problem

Consider the following simple model problem.

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega = \Omega_1 \cup \Omega_2 \\ u &= 0 \qquad \text{on } \Gamma_D = \partial\Omega\,. \end{aligned} \tag{6.1}$$

It's well known that the (weak-) Laplace problem (6.1) is uniquely solvable, because of

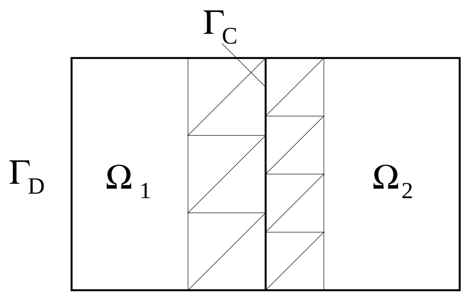


Figure 6.1: Non-matching grid

the $H_0^1(\Omega)$-ellipticity of the bilinear form $a(.,.)$ (*Poincaré* or *Friedrich* inequality), the boundness of $a(.,.)$ and the *Lax Milgram* lemma (Theorem 4.11). Due to the next theorem it's possible to handle *non-matching grids.*

Theorem 6.1. *The space $H_0^1(\Omega)$ can be represented as a subspace of V, namely*

$$H_0^1(\Omega) = \{v \in V \mid b(q,v) = 0 \quad \forall\, q \in Q\}\,, \tag{6.2}$$

with

$$\begin{aligned} V &:= H_{0,D}^1(\Omega_1) \times H_{0,D}^1(\Omega_2) \\ Q &:= \begin{cases} \left(H_{00}^{\frac{1}{2}}(\Gamma_C)\right)' & \overline{\Gamma_C} \cap \overline{\Gamma_D} = \partial\Gamma_C \\ H^{-\frac{1}{2}}(\Gamma_C) & \overline{\Gamma_C} \cap \overline{\Gamma_D} = \emptyset \end{cases} \\ b(q,v) &:= \langle q, [v] \rangle_{Q \times Q'} \\ [v] &:= (v_1 - v_2)|_{\Gamma_C}\,. \end{aligned}$$

Remark 6.2. To simplify the notation set

$$\begin{aligned} X &= L_2(\Gamma_C) \to X' = X \\ Y &= \begin{cases} H^{-1}(\Gamma_C) & \overline{\Gamma_C} \cap \overline{\Gamma_D} = \partial\Gamma_C \\ H_0^{-1}(\Gamma_C) & \overline{\Gamma_C} \cap \overline{\Gamma_D} = \emptyset \end{cases} \\ Z &= H^{-\frac{1}{2}}(\Gamma_C) \to Z' = H^{\frac{1}{2}}(\Gamma_C)\,. \end{aligned}$$

Proof. A more general proof is presented in Raviart and Thomas [41, Lemma 1].

$''\subseteq''$: Let $u \in H_0^1(\Omega)$ and from the trace theorem Theorem 3.5 it's know that $[u] = 0$ in Q' and thus $b(q,u) = \langle q, [u]\rangle_{Q\times Q'} = 0 \quad \forall\, q \in Q$.

$''\supseteq''$: Assume that there exists an element $v \in \{v \in V \,|\, b(q,v) = 0 \quad \forall\, q \in Q\}/H_0^1(\Omega)$. From the *Hahn-Banach* theorem (Heuser [31, Satz 42.5]) follows that

$$\exists\, v^* \in V' : \quad \langle\, v^*, v\rangle_{V'\times V} < \langle\, v^*, H_0^1(\Omega)\rangle_{V'\times V} .$$

Because $H_0^1(\Omega)$ is a linear space $\langle\, v^*, H_0^1(\Omega)\rangle_{V'\times V} = 0 \implies \langle\, v^*, v\rangle_{V'\times V} < 0$. The proof is done if $\exists\, q(v^*) \in Q\,\forall\, w \in V : \quad \langle\, v^*, w\rangle_{V'\times V} = \langle\, q(v^*), [w]\rangle_{Q\times Q'}$, because this would be a contradiction to the assumption. Also from the *Hahn-Banach* theorem it's known that (see Adams [1, Theorem 3.8])

$$\exists\,(g, g_0)(v^*) \in L_2(\Omega)^{d+1}\,\forall\, w \in V : \langle\, v^*, w\rangle_{V'\times V} = \sum_{l=1}^{2} \int_{\Omega_l} \langle\, g(v^*), \nabla w\rangle_{l_2(\mathbb{R}^d)} + g_0(v^*) w \, dx .$$

Note that $\forall\, \varphi \in C_0^\infty(\Omega) \subset H_0^1(\Omega)$

$$0 = \langle v^*, \varphi\rangle_{C'\times V} = \int_{\Omega} \langle\, g(v^*), \nabla\varphi\rangle_{l_2(\mathbb{R}^d)} + g_0(v^*)\varphi \, dx ,$$

and thus $\operatorname{div} g(v^*)$ exists in a distributional sense, i.e. $\operatorname{div} g(v^*) \in (C_0^\infty(\Omega))'$, and is equal to $g_0(v^*)$. From the fact that $C_0^\infty(\Omega)$ is dense in $L_2(\Omega)$ and that $g_0(v^*) \in L_2(\Omega)$ this equality $\operatorname{div} g(v^*) = g_0(v^*)$ holds also in $L_2(\Omega)$ and thus $g(v^*) \in H(\operatorname{div}, \Omega)$. Using this the linear functional v^* can be written as

$$\begin{aligned}
\langle\, v^*, w\rangle_{V'\times V} &= \sum_{l=1}^{2} \int_{\Omega_l} \langle\, g(v^*), \nabla w\rangle_{l_2(\mathbb{R}^d)} + \operatorname{div} g(v^*) w \, dx \\
&= \sum_{l=1}^{2} \int_{\Omega_l} \langle\, g(v^*), \nabla w\rangle_{l_2(\mathbb{R}^d)} + \operatorname{div} g(v^*) w \, dx \\
&\overset{\text{Lemma 3.8}}{=} \big\langle \langle\, g(v^*), n_1\rangle_{l_2(\mathbb{R}^d)}, w_1 \big\rangle_{Z\times Z'} + \big\langle \langle\, g(v^*), n_2\rangle_{l_2(\mathbb{R}^d)}, w_2 \big\rangle_{Z\times Z'} \\
&\overset{\langle [g(v^*)], n_l\rangle_{l_2(\mathbb{R}^d)} = 0}{=} \big\langle \langle\, g(v^*), n\rangle_{l_2(\mathbb{R}^d)}, [w] \big\rangle_{Z\times Z'} .
\end{aligned}$$

Note that $n = n_1 = -n_2$, $\langle\, g(v^*), n\rangle_{l_2(\mathbb{R}^d)} \in Z \subseteq Q$ and $[w] \in Q'$. With the setting $q(v^*) = \langle\, g(v^*), n\rangle_{l_2(\mathbb{R}^d)}$ the desired result

$$\langle v^*, w\rangle_{V'\times V} = \langle q(v^*), [w]\rangle_{Q\times Q'} = b(q(v^*), w) \qquad \forall\, w \in V$$

is proved.

□

Now consider the mixed *Laplace* problem

$$\begin{aligned} a(u,v) + b(p,v) &= \langle f, v\rangle_{V'\times V} && \forall\, v \in V \\ b(q,u) &= 0 && \forall\, q \in Q\,, \end{aligned} \tag{6.3}$$

where

$$a(u,v) \;:=\; \sum_{i=1}^{2} \int_{\Omega_i} \langle \nabla u, \nabla v\rangle_{l_2(\mathbb{R}^d)}\, dx\,.$$

The question arises if the problem (6.1) and problem (6.3) are equivalent. But first consider the question whether the bilinear form $b(.,.)$ fulfils the LBB-condition or not.

Remark 6.3. Because of Theorem 3.5 or Remark 3.7 the trace norm $\|.\|_{Q'}$ has an equivalent representation.

$$\|g\|_{Q'} \;\simeq\; \inf_{\substack{w\in V\\ [w]=g}} \|w\|_V \tag{6.4}$$

Note that V can be replaced by $V_j \times \{0\}$, because of $V_j|_{\Gamma_C} = Q'$.

Proposition 6.4. *Problem* (6.3) *fulfils the LBB condition*

$$\sup_{v\in V} \frac{b(q,v)}{\|v\|_V} \succeq \|q\|_Q \tag{6.5}$$

Proof. Because of the surjectivity $V_l|_{\Gamma_C} = Q'$ and $[V] = Q'$ the following estimates are valid.

$$\sup_{v\in V} \frac{b(q,v)}{\|v\|_V} \;\geq\; \sup_{v\in V_l\times\{0\}} \frac{b(q,v)}{\|v\|_V} \tag{6.6}$$

$$\sup_{v\in V_l\times\{0\}} \frac{b(q,v)}{\|v\|_V} \;=\; \sup_{g\in Q'} \sup_{\substack{w\in V\\ [w]=g}} \left\{ \frac{\langle q,g\rangle_{Q\times Q'}}{\|g\|_{Q'}} \, \frac{\|g\|_{Q'}}{\|w\|_V} \right\} \;\overset{\text{Remark 6.3}}{\succeq}\; \|q\|_Q$$

□

Proposition 6.5. *The mixed problem* (6.3) *has a unique solution* $(u,p) \in V \times Q$ *with*

$$\|u\|_V + \|p\|_Q \;\preceq\; \|f\|_{V'}$$

Proof. Because of Theorem 6.1 the bilinear form $a(.,.)$ is elliptic on $\ker B = \{v \mid b(v,q) = 0 \quad \forall\, q \in Q\}$. From Proposition 6.4 it's known that $b(.,.)$ fulfils a LBB-condition and with Theorem 4.25 the prove is done. □

The only thing which is missing is that a solution of (6.1) generates a solution of (6.3) and vice versa.

Theorem 6.6. *Let $u \in H_0^1(\Omega)$ be the unique solution of* (6.1) *with $f \in L_2(\Omega)$ then the pair $(u, \frac{\partial u}{\partial n}) \in V \times Z$ is the unique solution of* (6.3). *On the other hand if $(u,p) \in V \times Q$ is the unique solution of* (6.3) *then $u \in H_0^1(\Omega)$ is the unique one of* (6.1).

Proof. The proof can also be found in RAVIART AND THOMAS [41, Theorem 1].

- Let $(u,p) \in V \times Q$ be the unique solution of (6.3), then due to Theorem 6.1 $u \in H_0^1(\Omega)$. Choosing $v \in H_0^1(\Omega)$ in equation (6.3) the desired result follows.

- Let u be the solution of (6.1) and consider the linear form

$$\langle L, v\rangle_{V' \times V} := a(u,v) - \langle f, v\rangle_{V' \times V} = b(p,v)$$

Let $\varphi \in C_0^\infty(\Omega) \subset H_0^1(\Omega)$ and because u is a solution of (6.1) it's deduced that $-\mathrm{div}\,\nabla u$ exists in a distributional sense and is equal f.

$$0 = \langle L, \varphi\rangle_{V' \times V} = \int_\Omega \langle \nabla u, \nabla \varphi\rangle_{l_2(\mathbb{R}^d)} - f\varphi\, dx = -\langle \mathrm{div}\,\nabla u + f, \varphi\rangle_{\left(C_0^\infty(\Omega)\right)' \times C_0^\infty(\Omega)}$$

From the density of $C_0^\infty(\Omega)$ in $L_2(\Omega)$ and the fact that $f \in L_2(\Omega)$ the equality holds in $L_2(\Omega)$ and thus $\mathrm{div}\,\nabla u \in H(\mathrm{div}, \Omega)$. Using *Greens'* formula (Lemma 3.8), the linear functional $\langle L, .\rangle_{V' \times V}$ can be represented as

$$\begin{aligned}\langle L, v\rangle_{V' \times V} &= -\langle \mathrm{div}\,\nabla u + f, v\rangle_{V' \times V} + \langle\, \langle \nabla u, n\rangle_{l_2(\mathbb{R}^d)}, [v]\rangle_{Z \times Z'} \\ &= b(\langle \nabla u, n\rangle_{l_2(\mathbb{R}^d)}, v)\end{aligned}$$

From Proposition 6.4 it's known that p is unique and thus $p = \langle \nabla u, n\rangle_{l_2(\mathbb{R}^d)} \in Z$.

□

Remark 6.7. The mixed formulation is useful if Ω_l are separately meshed because in this case the grid is usually a non-matching one (see Figure 6.1).

Remark 6.8. To solve this problem numerically, a stable pair $V_h \subset V$, $Q_h \subset Q$ of finite element spaces has to be chosen. For V assume the standard discretisation with *Lagrange* elements of first order. The only thing which is missing is how to choose Q_h. One possibility is to use *Mortar*, which is a well developed technique. An other possibility is to use *nodal constraints*, which will be considered in the following.

6.2 Nodal Constraints

The idea of *nodal constraints* is to set the duality pairing

$$\langle q_h, [u_h]\rangle_{Q \times Q'} = \langle q_h, u_{h,1} - u_{h,2}\rangle_{Q \times Q'} = \sum_{x_i \in \mathcal{N}_{h,l}^C} q_h^i(u_{h,1}(x_i) - u_{h,2}(x_i)),$$

where $q_h^i \in \mathbb{R}$ and $\mathcal{N}_{h,l}^C = \{\text{nodes of surface } \Gamma_C \text{ generated by the mesh of } \Omega_l\}$. Now the question arises are there linear functionals $q_{h,l}^i : Q' \to \mathbb{R} \in Q$ such that

$$\langle q_{h,l}^i, v_h \rangle_{Q \times Q'} = v_h(x_i) \qquad \forall\, v_h \in W_h \tag{6.7}$$

or not. The space W_h denotes the sum of the traces of $V_{h,l}$ at Γ_C (see Figure 6.2).

$$W_h := \operatorname{tr} V_{h,1} + \operatorname{tr} V_{h,2} \,. \tag{6.8}$$

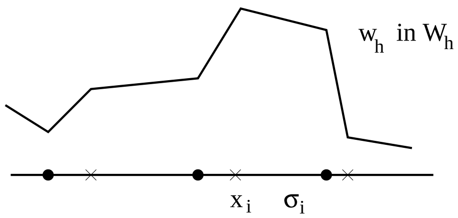


Figure 6.2: Summed trace spaces

The answer isn't obvious because in general the space Q doesn't include point evaluating functionals. Having a positive answer, the next question is whether the space $Q_{h,l} = \operatorname{span}\{q_{h,l}^i \,|\, x_i \in \mathcal{N}_{h,l}^C\}$ and V_h are a stable pairing, or not. This section is concerned with answering this questions by formulating an abstract condition for the existence of an interpolation operator $I^l : \operatorname{tr}V_l \to \operatorname{tr}V_{h,l}$ such that

$$I^l(v_h)(x_i) = v_h(x_i) \qquad \forall\, x_i \in \mathcal{N}_{h,l}^C \,\forall\, v_h \in \operatorname{tr}V_{h,1} + \operatorname{tr}V_{h,2} \tag{6.9a}$$

$$I^l v_h = v_h \qquad \forall\, v_h \in \operatorname{tr}V_{h,l} \tag{6.9b}$$

$$\|I^l v\|_{k,E} \preceq \sum_{j=0}^{m} h_E^{j-k} |v|_{j,\omega_E} \qquad \forall\, 0 \le k \le m \,\forall\, v \in H^m(\Gamma_C) \tag{6.9c}$$

$$\sum_{E \in \mathcal{E}_{h,l}^C} h_E^{2(k-m)} \|v - I^l v\|_{k,E}^2 \preceq \|v\|_{m,\Gamma_C}^2 \qquad \forall\, 0 \le k \le m \le r+1 \,\forall\, v \in H^m(\Gamma_C)\,. \tag{6.9d}$$

r is the integer which fixes the polynomial degree of the ansatz space (for linear Lagrange elements $r = 1$).

Remark 6.9. W.l.o.g. it's possible to refer $I^l : Q' \to \operatorname{tr}V_{h,l}$ as an operator from $I^l : V_l \to V_{h,l}$ with property (6.9a)-(6.9d) on V_l, $V_{h,l}$ instead of $\operatorname{tr}V_l$, $\operatorname{tr}V_{h,l}$ and Ω_l instead of Γ_C. The reason therefore is

- $\operatorname{tr}V_l = Q'$ because the trace operator is surjective Theorem 3.5
- $I^l : V_l \to V_{h,l}$ can be constructed like a SCOTT-ZHANG operator using instead of the usual boundary approximation the operator $I^l : \operatorname{tr}V_l \to \operatorname{tr}V_{h,l}$.

Indeed $I^l : V_l \to V_{h,l}$ becomes nothing else than a SCOTT-ZHANG operator with special choice of σ_i for the boundary nodes $x_i \in \mathcal{N}^C_{h,l}$.

With this knowledge it's possible to refer $I_h : V \to V_h \ (v_1, v_2) \mapsto (I^1 v_1, I^2 v_2)$ as an interpolation operator with property (6.9b)- (6.9d).

Remark 6.10. From Section 5.2 it's known that $I^l : V_l \to V_{h,l}$ can be represented via $I^l(v) = \sum_{x_i \in \mathcal{N}_{h,l}} I^l(v)(x_i)\varphi_i$, where $\varphi_{i,l}$ denotes the basis ansatz function in node x_i. Similar the interpolation operator $I^l : \mathrm{tr}V_l \to \mathrm{tr}V_{h,l}$ is represented via $I^l(v) = \sum_{x_i \in \mathcal{N}^C_{h,l}} I^l(v)(x_i)\varphi_{i,l}$, where $\forall\, x_i \in \mathcal{N}^C_{h,l} : \quad \varphi_{i,l} := \varphi_i|_{\Gamma_C}$.

Assuming the existence of such an interpolation operator fulfilling (6.9a)-(6.9d) the space of nodal evaluating linear functionals $Q_{h,l}$ can be defined.

Definition 6.11. *The space of nodal evaluating functionals is defined by*

$$Q_{h,l} = \left\{ q \in Q \ \mid \ q(v) = \sum_{x_i \in \mathcal{N}^C_{h,l}} q_i\,(I^l v)(x_i) \quad q_i \in \mathbb{R} \right\} . \tag{6.10}$$

Remark 6.12. To call the space $Q_{h,l}$ the space of nodal evaluating functionals isn't correct because only functions $v \in W_h$ are evaluated correct at the nodal points $x_i \in \mathcal{N}_{h,l}$. Even continuous functions aren't evaluated at node x_i.

Considering $Q_{h,l}$ as discretisation of Q and with the discretisation $V_h \subset V$ the mixed FE system

$$\begin{aligned} a(u_h, v_h) + b(p_h, v_h) &= \langle f, v_h \rangle_{V' \times V} && \forall\, v_h \in V_h \\ b(q_h, u_h) &= 0 && \forall\, q_h \in Q_{h,l} \end{aligned} \tag{6.11}$$

can be considered. To prove existence, uniqueness and approximation results for the mixed system (6.11) standard assumptions for Theorem 4.25 and Theorem 5.10 have to be verified. First consider the approximation property of $Q_{h,l}$ in $Q \cap X$ ($X = L_2(\Gamma_C)$).

Lemma 6.13. *Let I^l fulfil properties* (6.9a)- (6.9d)*, $Q_{h,l}$ be given by Definition 6.11 and assume a uniform mesh. Then the operator*

$$I^l_Q : Q \to Q_{h,l} \quad p \mapsto p_h := \sum_{x_i \in \mathcal{N}^C_{h,l}} \langle p, \varphi_{i,l} \rangle_{Y \times Y'} I^l(.)(x_i) \,. \tag{6.12}$$

is an interpolation operator fulfilling

$$\begin{aligned} \|I^l_Q p\|_Q &\preceq \|p\|_Q && \forall\, p \in Q \\ \|p - I^l_Q\|_X &\preceq h_l^{\frac{1}{2}} \|p\|_Q && \forall\, p \in X \,. \end{aligned} \tag{6.13}$$

Proof. Because of the interpolation theorem (Theorem 3.13) Q is the interpolation space $Q = [Y, X]_{\frac{1}{2}}$. To hold the notation simple define $Q_\theta := [Y, X]_\theta$. Note that $X \hookrightarrow Y$ and thus $\langle p, q^*\rangle_{Q_\theta \times Q'_\theta} = \langle p, q^*\rangle_{Y\times Y'}$. From the definition of I^l_Q follows that

$$\langle I^l_Q p, q^*\rangle_{Q_\theta\times Q'_\theta} = \langle p, \sum_{x_i\in\mathcal{N}^C_{h,l}} I^l(q^*)(x_i)\varphi_{i,l}\rangle_{Y\times Y'} = \langle p, I^l q^*\rangle_{Y\times Y'} = \langle p, I^l q^*\rangle_{Q_\theta\times Q'_\theta} \,.$$

- That I^l_Q is bounded is proved by proofing that I^l_Q is a bounded linear operator from $I^l_Q : Q_\theta \to (Q_{h,l}, \|.\|_{Q_\theta})$ for $\theta \in \{0, 1\}$ and then using Theorem 3.11.

$$\begin{aligned} \|I^l_Q p\|_{Q_\theta} &= \sup_{q^*\in Q'_\theta} \frac{\langle I^l_Q p, q^*\rangle_{Q_\theta\times Q'_\theta}}{\|q^*\|_{Q'_\theta}} = \sup_{q^*\in Q'_\theta} \frac{\langle p, I^l q^*\rangle_{Q_\theta\times Q'_\theta}}{\|q^*\|_{Q'_\theta}} \\ &\overset{C.S.}{\leq} \|p\|_{Q_\theta} \sup_{q^*\in Q'_\theta} \frac{\|I^l q^*\|_{Q'_\theta}}{\|q^*\|_{Q'_\theta}} \overset{(6.9c)}{\preceq} \|p\|_{Q_\theta} \end{aligned}$$

- Also the approximation property is proved by proofing the corresponding approximation property in the space Q_θ with $\theta \in \{0, 1\}$ and using the interpolation inequality (Theorem 3.13).

$$\begin{aligned} \|p - I^l_Q p\|_{Q_\theta} &= \sup_{q^*\in Q'_\theta} \frac{\langle p - I^l_Q p, q^*\rangle_{Q_\theta\times Q'_\theta}}{\|q^*\|_{Q'_\theta}} = \sup_{q^*\in Q'_\theta} \frac{\langle p, q^* - I^l q^*\rangle_{X\times X}}{\|q^*\|_{Q'_\theta}} \\ \Rightarrow \|p - I^l_Q p\|_Y &\overset{(6.9d)}{\preceq} \sqrt{\sum_{E\in\mathcal{E}^C_{h,l}} h_E^2 \|p\|^2_{0,E}} \leq h_l \|p\|_X \\ \Rightarrow \|p - I^l_Q p\|_{Q_\theta} &\overset{\text{Theorem 3.13}}{\preceq} \|p - I^l_Q p\|^\theta_Y \|p - I^l_Q p\|^{1-\theta}_X \preceq h^\theta_l \|p\|_X \end{aligned}$$

□

A direct consequence of Lemma 6.13 is the following proposition.

Proposition 6.14. *Let I^l fulfil properties* (6.9a)- (6.9d)*, $Q_{h,l}$ be given by Definition 6.11 and assume a uniform mesh. Assuming $p \in X$, then the following approximation result is valid.*

$$\inf_{q_h\in Q_{h,l}} \|p - q_h\|_Q \preceq \sqrt{h_l}\|p\|_X \tag{6.14}$$

Proposition 6.15. *The discretised problem* (6.11) *fulfils the discrete LBB condition uniformly in h_l.*

Proof. The proof is done by proving that I_h is a Fortin operator on the subspace $V_l \times \{0\}$ and thus for equation (6.6). Using *Fortin's criterion* (Theorem 5.4) everything is done. Boundness of I_h is valid because of requirement (6.9c). The only thing to prove is

$$\langle q_h, v - I^l v\rangle_{Q\times Q'} = 0 \qquad \forall\, q_h \in Q_{h,l}\, \forall\, v \in V_l \times \{0\}\,.$$

This is trivial because $\forall\, v \in V_l \times \{0\}: \quad I^l(v) = I^l(I_h v)$ and thus

$$\langle q_h, v\rangle_{Q\times Q'} = \sum_{x_i\in\mathcal{N}_{h,l}^C} q_h^i I^l(v)(x_i) \overset{(6.9b)}{=} \sum_{x_i\in\mathcal{N}_{h,l}^C} q_h^i I^l(I_h(v))(x_i) = \langle q_h, I_h v\rangle_{Q\times Q'}\,. \tag{6.15}$$

□

Proposition 6.16. *The bilinear form $a(.,.)$ is uniformly V_{0h}-elliptic, i.e.*

$$\|u\|_V^2 \preceq a(u,u) \qquad \forall\, u \in ker B_{h,l} = \{v \in V_h \,|\, b(q_h, v) = 0 \quad \forall\, q_h \in Q_{h,l}\}\,. \tag{6.16}$$

Proof.

- $\overline{\Gamma_C} \cap \overline{\Gamma_D} = \partial\Gamma_C$: In this case $|\partial\Omega_j \cap \Gamma_D| > 0 \quad \forall\, j \in \{1,2\}$ and thus the *Poincaré* (or *Friedrich's*) inequality is valid on all of V, i.e. $a(.,.)$ is V-elliptic.
- $\overline{\Gamma_C} \cap \overline{\Gamma_D} = \emptyset$: W.l.o.g. $|\partial\Omega_2 \cap \Gamma_D| = 0$ and thus $a(.,.)$ isn't V-elliptic. Thus prove that $a(.,.) + \|\,[\,.\,]\,\|_{L_2(\Gamma_C)}^2 \simeq \|.\|_V^2$. This is done by verifying the assumptions of Theorem 3.18 (equivalent norms theorem), i.e. $\|[\,.\,]\|_{L_2(\Gamma_C)} \le \|.\|_V$ and continuous, and $\forall\, c \in V\; (c = \text{const.}): \|\,[c]\,\|_{L_2(\Gamma_C)} = 0 \Leftrightarrow c = 0$. Both assumptions are trivially fulfilled. Now consider $I_h u \in \ker B_{h,l} \Leftrightarrow I^l[I_h u] = 0$. For such functions the term $\|\,[I_h u]\,\|_{L_2(\Gamma_C)}$ doesn't vanish in general. Nevertheless it's possible to estimate

$$\begin{aligned}
\|\,[I_h u]\,\|_{L_2(\Gamma_C)}^2 &= \|\,[I_h I_h u] - I^l[I_h u]\,\|_{L_2(\Gamma_C)}^2 = \|(I^l - I^{l-1})(I_h u)_{l-1}\|_{L_2(\Gamma_C)}^2 \\
&\le \|(I - I^l)(I_h u)_{l-1}\|_{L_2(\Gamma_C)}^2 + \|(I - I^{l-1})(I_h u)_{l-1}\|_{L_2(\Gamma_C)}^2 \\
&\overset{(6.9d)}{\preceq} \sum_{E\in\mathcal{E}_{h,l}^C} h_E \|(I_h u)_{l-1}\|_{\frac{1}{2},E}^2 \overset{(6.4)}{\preceq} h_l \|I_h u\|_V^2\,.
\end{aligned}$$

  Thus there exists a parameter $h_{l,\max} \in \mathbb{R}^+$ such that for all mesh-parameters $h_l \le h_{l,\max}$ the bilinear form $a(.,.)$ is uniform V_{0h}-elliptic.

Note that for every mesh-parameter $a(.,.)$ is trivially V_{0h}-elliptic (but not necessary uniformly) because $a(.,.) + \|I^l[\,.\,]\,\|_{L_2(\Gamma_C)}^2 \simeq_h \|.\|_V^2$ (I^l is uniformly bounded and $\|I^l[c]\,\| = |\Gamma_C|\,|\,[c]\,|$ and thus Theorem 3.18 is valid). Per definition $\forall\, I^h u \in \ker B_{h,l}: \|I^l[I^h u]\|_{L_2(\Gamma_C)} = 0$ and everything is done. □

Theorem 6.17. *Let $(u,p) \in ((H^2(\Omega_1) \times H^2(\Omega_2)) \cap V) \times L_2(\Gamma_C)$ be the solution of problem* (6.3) *and (u_h, p_h) be solution of* (6.11)*, with a uniform mesh, then the following convergence result holds*

$$\|u - u_h\|_V + \|p - p_h\|_Q \preceq h_l\,\|u\|_{2,\Omega_1+\Omega_2} + h_l^{\frac{1}{2}}\,\|p\|_X \tag{6.17}$$

Proof. Because of the discrete inf-sup condition (Proposition 6.15) the V_{0h} ellipticity of $a(.,.)$ (Proposition 6.16) and the boundness, Theorem 5.10 is valid and thus

$$\begin{aligned}
\|u - u_h\|_V + \|p - p_h\|_Q &\preceq \inf_{v_h\in V_h} \|u - v_h\|_V + \inf_{q_h\in Q_{h,l}} \|p - q_h\|_Q \\
&\overset{\text{Proposition 6.14}}{\preceq} \|u - I_h u\|_V + \sqrt{h_l}\|p\|_{0,\Gamma_C} \\
&\overset{(6.9d)}{\preceq} h_l\|u\|_{2,\Omega_1+\Omega_2} + \sqrt{h_l}\|p\|_{0,\Gamma_C}\,.
\end{aligned}$$

□

Remark 6.18. The convergence result of Theorem 6.17 is an optimal result for the dual variable p and the assumed smoothness. But in the usual case that $u \in H^2(\Omega) \cap H_0^1(\Omega)$, then $p \in H^{\frac{1}{2}}(\Gamma_C)$ due to Theorem 6.6 and thus a better convergence result is expected. For *Mortar* techniques this is true but for *nodal constraints* this can't be proved because of missing approximation results for negative sobolev indices.

Nevertheless it's possible to achieve a better convergence result for the primal variable u, if the mixed *Laplace* problem is consistently modified. How to modify the mixed *Laplace* problem consistently such that the convergence result is optimal for the primal variable u and for *nodal constraints*, is the aim of the next section.

6.3 Stabilised Mixed FEM

In Section 6.2 a master surface l was chosen and on the master surface l the nodal interpolation functions were defined. There may be the idea to expect the nodal restrictions for both sides (both surfaces). In general this isn't a good idea because either the new kernel space

$$\ker B_h = \{v \in V_h \,|\, b(q_{h,l}, v) = 0 \,\wedge\, b(q_{h,l-1}, v) = 0 \quad \forall\, q_{h,l} \in Q_{h,l} \,\forall\, q_{h,l-1} \in Q_{h,l-1}\}$$

is equal to the set of functions which are constant on Γ_C or the LBB condition isn't fulfilled or both on some parts of Γ_C. The first case is too restrictive and the solution will in general not converge to the continuous one, in the second case the discrete solution isn't unique.

Nevertheless the hanging nodes of the slave surface $l-1$ (i.e. nodes which are not restricted), may influence the convergence property. For the *Mortar* method there are no hanging nodes. The idea now is to add some consistent penalty term to the bilinear form $a(.,.)$ which restricts hanging nodes of the slave surface $l-1$. The simplest idea is to add some penalty term for every node at the slave surface $l-1$. Note that adding a penalty term isn't the same as expecting nodal constraints on both sides.

For notation assume that the master-surface is the surface $l-1$ (i.e. $q_h \in Q_{h,l-1}$) and the slave surface is l. Consider the following consistent modified bilinear form

$$a_{l,s}(u,v) = a(u,v) + \sum_{E \in \mathcal{E}_{h,l}^C} h_E^{-2\alpha} \int_E I^l[u]\, I^l[v]\, ds\,. \tag{6.18}$$

The penalty term is h_l-dependent and penalises every node of the slave surface. To apply standard theory, i.e. boundness, V_{0h}-ellipticity and LBB condition has to be verified, new h_l dependent norms have to be introduced (at least for the primal variable).

$$\|v\|_{V_{h,l}}^2 \;:=\; \|v\|_V^2 + \sum_{E \in \mathcal{E}_{h,l}^C} h_E^{-2\alpha} \|u_1 - u_2\|_{0,E}^2 \tag{6.19}$$

$$\|q\|_{Q_{h,l}}^2 \;:=\; \inf_{\substack{q = q_1 + q_2 \\ q_1 \in Q,\, q_2 \in X}} \{\|q_1\|_Q^2 + \sum_{E \in \mathcal{E}_{h,l}^C} h_E^{2\alpha} \|q_2\|_{0,E}^2\}\,. \tag{6.20}$$

Remark 6.19. The h_l dependent norm $\|.\|_{V_{h,l}}$ is stronger than the former norm $\|.\|_V$ but the corresponding h_l dependent norm of the dual variable $\|.\|_{Q_{h,l}}$ is weaker than $\|.\|_Q$. Important is that for $p \in X(= L_2(\Gamma_C))$

$$\inf_{q_h \in Q_{h,l}} \|p - q_h\|^2_{Q_{h,l}} \leq \sum_{E \in \mathcal{E}^C_{h,l}} h_E^{2\alpha} \|p\|^2_{0,E} \,.$$

I.e. the h_l dependent norm $\|.\|_{Q_{h,l}}$ is chosen weaker than $\|.\|_Q$ such that the approximation of the dual variable is trivial. Especially for $\alpha \geq \frac{1}{2}$ the approximation result improves for the dual variable.
Note that this norms are not well suited for numerical implementations but that's not important because they are only needed for theoretical aspects.

Thus the aim is to check the properties of Theorem 4.25 and Theorem 5.10. To do this the explicite representation of the dual norm of $\|.\|_{Q_{h,l}}$ is needed.

Lemma 6.20. *The dual norm of* $\|.\|_{Q_{h,l}}$ *is*

$$\|\cdot\|^2_{Q'_{h,l}} = \|\cdot\|^2_{Q'} + \sum_{E \in \mathcal{E}^C_{h,l}} h_E^{-2\alpha} \|\cdot\|^2_{0,E} \,. \tag{6.21}$$

Proof. Let $q^* \in Q'$, then the norm $\|q^*\|_{Q'_{h,l}}$ is

$$\begin{aligned}
\|q^*\|_{Q'_{h,l}} &= \sup_{q \in Q} \frac{\langle q^*, q\rangle_{Q\times Q'}}{\|q\|_{Q_{h,l}}} = \sup_{q\in Q} \sup_{\substack{q = q_1 + q_2 \\ q_1 \in Q,\, q_2 \in X}} \frac{\langle q^*, q_1 + q_2\rangle_{Q\times Q'}}{\sqrt{\|q_1\|^2_Q + \sum_{E \in \mathcal{E}^C_{h,l}} h_E^{2\alpha}\|q_2\|^2_{0,E}}} \\
&= \sup_{\substack{q = q_1 + q_2 \\ q_1 \in Q,\, q_2 \in X}} \frac{\langle q^*, q_1 + q_2\rangle_{Q\times Q'}}{\sqrt{\|q_1\|^2_Q + \sum_{E \in \mathcal{E}^C_{h,l}} h_E^{2\alpha}\|q_2\|^2_{0,E}}} \\
&\overset{C.S.}{\leq} \sup_{\substack{q = q_1 + q_2 \\ q_1 \in Q,\, q_2 \in X}} \frac{\|q^*\|_{Q'}\|q_1\|_Q + \sqrt{\sum_{E \in \mathcal{E}^C_{h,l}} h_E^{-2\alpha}\|q^*\|^2_{0,E}}\sqrt{\sum_{E \in \mathcal{E}^C_{h,l}} h_E^{2\alpha}\|q_2\|^2_{0,E}}}{\sqrt{\|q_1\|^2_Q + \sum_{E \in \mathcal{E}^C_{h,l}} h_E^{2\alpha}\|q_2\|^2_{0,E}}} \\
&\overset{C.S.}{\leq} \sqrt{\|q^*\|_{Q'} + \sum_{E \in \mathcal{E}^C_{h,l}} h_E^{-2\alpha}\|q^*\|^2_{0,E}} \,.
\end{aligned}$$

The equality is proved by considering the solution of the following variational equation

$$\langle p_1, q_1\rangle_{Q\times Q} + \sum_{E \in \mathcal{E}^C_{h,l}} h_E^{2\alpha}\langle p_2, q_2\rangle_{0,E} = \langle q^*, q_1 + q_2\rangle_{Q\times Q'} \qquad \forall\, q_1 \in Q\, \forall\, q_2 \in X \,.$$

Note that Q is a Hilbert space and $\langle .,.\rangle_{Q\times Q}$ denotes the scalar product on Q and not the duality pairing. This variational equality is uniquely solvable because it's $Q \times L_2(\Gamma_C)$-elliptic and bounded (not uniformly in h_l) and thus the theorem of *Lax Milgram* (Theorem 4.11) is valid. Taking the supremum of the variational equation and using that $L_2(E) \subseteq X(= L_2(\Gamma_C)) \quad \forall\, E \in \mathcal{E}^C_{h,l}$ it can be concluded that (p_1, p_2) fulfils

$$\|p_1\|_Q = \|q^*\|_{Q'} \qquad \|p_2\|^2_X = \sum_{E\in\mathcal{E}^C_{h,l}} h_E^{-2\alpha}\|q^*\|^2_{0,E}\,.$$

Substituting this result into above supremum the desired lower bound follows. ☐

Remark 6.21. Note that due to Remark 6.3 it's possible to give an equivalent representation of $\|.\|_{Q'_{h,l}}$.

$$\begin{aligned}\|g\|^2_{Q'_{h,l}} &= \|g\|^2_{Q'} + \sum_{E\in\mathcal{E}^C_{h,l}} h_E^{-2\alpha}\|g\|^2_{0,E} \\ &\overset{(6.4)}{\simeq} \inf_{\substack{w\in V\\ [w]=g}} \left(\|w\|^2_V + \sum_{E\in\mathcal{E}^C_{h,l}} h_E^{-2\alpha}\|\,[w]\,\|^2_{0,E} \right) \simeq \inf_{\substack{w\in V\\ [w]=g}} \|w\|^2_{V_{h,l}}\,.\end{aligned} \tag{6.22}$$

In this equivalent representation V can be replaced by $V_j \times \{0\}$, because $V_j|_{\Gamma_C} = Q'$. This equivalence was also valid for $\|.\|_{Q'}$ (Remark 6.3) and played an essential role for the LBB condition (see Proposition 6.4).

To apply standard theory the continuity, the LBB-condition and the kernel ellipticity for problem (6.3) with the stabilised bilinear form $a_{l,s}(.,.)$ ((6.18)) instead of $a(.,.)$ is proved.

Proposition 6.22. *$a_{l,s}(.,.)$ is continuous and V_0-elliptic, $b(.,.)$ is continuous and fulfils the LBB condition with respect to the h_l-dependent norms $\|.\|_{V_{h,l}}$, $\|.\|_{Q_{h,l}}$.*

Proof.
- Continuity of $a_{l,s}(.,.)$:

$$\begin{aligned} a_{l,s}(u,v) &= a(u,v) + \sum_{E\in\mathcal{E}^C_{h,l}} h_E^{-2\alpha} \int_E I^l[u]\, I^l[v]\, ds \\ &\overset{C.S.}{\preceq} \|u\|_V\|v\|_V + \sum_{E\in\mathcal{E}^C_{h,l}} h_E^{-2\alpha}\|I^l[u]\,\|_{0,E}\|I^l[v]\,\|_{0,E} \\ &\overset{C.S.+(6.9c)}{\preceq} \sqrt{\|u\|^2_V + \sum_{E\in\mathcal{E}^C_{h,l}} h_E^{-2\alpha}\|\,[u]\,\|^2_{0,E}}\sqrt{\|v\|^2_V + \sum_{E\in\mathcal{E}^C_{h,l}} h_E^{-2\alpha}\|\,[v]\,\|^2_{0,E}} \\ &= \|u\|_{V_{h,l}}\|v\|_{V_{h,l}} \end{aligned}$$

- Continuity of $b(.,.)$:

$$\begin{aligned} b(q,u) &= \langle q,[u]\rangle_{Q\times Q'} = \inf_{\substack{q=q_1+q_2\\ q_1\in Q,\,q_2\in X}} \langle q_1+q_2,[u]\rangle_{Q\times Q'} \\ &\overset{C.S.}{\leq} \inf_{\substack{q=q_1+q_2\\ q_1\in Q,\,q_2\in X}} \left\{ \|q_1\|_Q \|\,[u]\,\|_{Q'} + \sum_{E\in\mathcal{E}^C_{h,l}} h_E^{\alpha}\|q_2\|_{0,E}\, h_E^{-\alpha}\|\,[u]\,\|_{0,E} \right\} \\ &\overset{C.S.}{\leq} \inf_{\substack{q=q_1+q_2\\ q_1\in Q,\,q_2\in X}} \sqrt{\|q_1\|_Q^2 + \sum_{E\in\mathcal{E}^C_{h,l}} h_E^{2\alpha}\|q_2\|_{0,E}^2} \times \\ &\qquad \sqrt{\underbrace{\|\,[u]\,\|_{Q'}^2}_{\simeq\|u\|_V^2} + \sum_{E\in\mathcal{E}^C_{h,l}} h_E^{-2\alpha}\|\,[u]\,\|_{0,E}^2} \overset{(6.22)}{\preceq} \|q\|_{Q_{h,l}}\|u\|_{V_{h,l}} \end{aligned}$$

- LBB-condition:

$$\sup_{v\in V}\frac{b(q,v)}{\|v\|_{V_{h,l}}} \geq \sup_{v\in V_l\times\{0\}}\frac{b(q,v)}{\|v\|_{V_{h,l}}} \tag{6.23}$$

$$\begin{aligned} \sup_{v\in V_l\times\{0\}}\frac{b(q,v)}{\|v\|_{V_{h,l}}} &= \sup_{g\in Q'}\sup_{\substack{w\in V\\ [w]=g}} \left\{ \frac{\langle q,g\rangle_{Q\times Q'}}{\|g\|_{Q'}} \frac{\|g\|_{Q'}}{\|w\|_{V_{h,l}}} \right\} \\ &\overset{\text{Remark 6.21}}{\succeq} \sup_{g\in Q'}\frac{\langle q,g\rangle_{Q\times Q'}}{\|g\|_{Q'_{h,l}}} = \|q\|_{Q_{h,l}}\,. \end{aligned}$$

- V_0- ellipticity is trivial because $[u]=0\,\text{in}\,Q'$ and thus

$$a_{l,s}(u,u) = a(u,u) \succeq \|u\|_V^2 = \|u\|_{V_{h,l}}^2$$

□

In the same way as in Section 6.2 the uniform LBB-condition, for the h_l dependent norms $\|.\|_{V_{h,l}}$, $\|.\|_{Q_{h,l}}$, can be proved.

Proposition 6.23. *The bilinear form $b(.,.)$ fulfils the discrete LBB condition, with respect to the h_l dependent norms $\|.\|_{V_{h,l}}$, $\|.\|_{Q_{h,l}}$, uniformly in h_l.*

Proof. The prove is done similar as the prove of Proposition 6.15. It's shown that I_h is a *Fortin operator* on $V_l\times\{0\}$, i.e. that equation (6.23) fulfils the uniform LBB-condition with the associated norm $\|.\|_{V_{h,l}}$, $\|.\|_{Q_{h,l}}$. The invariance was still proved in Proposition 6.15 equation (6.15) and thus only the continuity of I_h with respect to the norm $\|.\|_{V_{h,l}}$ has to be verified.

$$\begin{aligned} \|I_hv\|_{V_{h,l}}^2 &= \|I_hv\|_V^2 + \sum_{E\in\mathcal{E}^C_{h,l}} h_E^{-2\alpha}\|\,[I_hv]\,\|_{0,E}^2 \\ &\overset{(6.9d)}{\preceq} \|v\|_V^2 + \sum_{E\in\mathcal{E}^C_{h,l}} h_E^{-2\alpha}\|\,[v]\,\|_{0,E}^2 = \|v\|_{V_{h,l}}\,. \end{aligned}$$

The rest follows from *Fortin's criterion* Theorem 5.4. □

The only result which is missing to apply standard convergence results for saddle point problems is the uniform V_{0h}-ellipticity with respect to the norm $\|.\|_{V_{h,l}}$. The author isn't in the position to give an general proof for the V_{0h}-ellipticity and thus it's assumed. In Section 6.4 a proof for linear Lagrange elements and the $d = 2$ is given.

Theorem 6.24. *Assume that $a_{l,s}(.,.)$ is uniform V_{0h}-elliptic, with respect to the norm $\|.\|_{V_{h,l}}$, that the solution (u,p) of (6.3) has the regularity $u \in H^2(\Omega_1) \times H^2(\Omega_2) \cap V$, $u|_{\Gamma_C} \in H^2(\Gamma_C)$, $p \in L_2(\Gamma_C)$ and let (u_h, p_h) be the solution of the discretised system (6.11), with $a_{l,s}(.,.)$ instead of $a(.,.)$, then the following a-priori error estimate is valid*

$$\begin{aligned} \|u - u_h\|_{V_{h,l}} + \|p - p_h\|_{Q_{h,l}} &\preceq \inf_{\substack{v_h \in V_h \\ q_h \in Q_{h,l}}} \{\|u - v_h\|_{V_{h,l}} + \|p - q_h\|_{Q_{h,l}}\} \\ &\preceq \sqrt{\sum_{T\in\tau_h} h_T^2 \|u\|_{2,T}^2 + \sum_{E\in\mathcal{E}_{h,l}^C} h_E^{2(2-\alpha)} \|u\|_{2,E}^2 + \sum_{E\in\mathcal{E}_{h,l}^C} h_E^{2\alpha} \|p\|_{0,E}^2} \quad . \end{aligned} \tag{6.24}$$

Proof. Because of Proposition 6.22 for the continuous problem and Proposition 6.23 and the assumed uniform V_{0h}-ellipticity, with respect to the norm $\|.\|_{V_{h,l}}$, for the discrete problem, all assumptions for Theorem 5.10 are valid and thus

$$\begin{aligned} &\|u - u_h\|_{V_{h,l}} + \|p - p_h\|_{Q_{h,l}} \\ &\quad\preceq \inf_{\substack{v_h \in V_h \\ q_h \in Q_{h,l}}} \{\|u - v_h\|_{V_{h,l}} + \|p - q_h\|_{Q_{h,l}}\} \\ &\quad\leq \sqrt{\|u - I_h u\|_V^2 + \sum_{E\in\mathcal{E}_{h,l}^C} h_E^{-2\alpha} \|u - I_h u\|_{0,E}^2} + \|p\|_{Q_{h,l}} \\ &\quad\overset{(6.9d)}{\preceq} \sqrt{\sum_{T\in\tau_h} h_T^2 \|u\|_{2,T}^2 + \sum_{E\in\mathcal{E}_{h,l}^C} h_E^{2(1-\alpha)} \|u\|_{2,E}^2 + \sum_{E\in\mathcal{E}_{h,l}^C} h_E^{2\alpha} \|p\|_{0,E}^2} \end{aligned}$$

□

Remark 6.25. The best choice for α, to achieve a best possible convergence result, is $\alpha = 1$. For a uniform mesh the convergence error in the primal variable u gets $\|u - u_h\|_{V_{h,l}} \preceq h(\|u\|_{2,\Omega_1} + \|u\|_{2,\Omega_2} + \|p\|_{L_2(\Gamma_C)})$ and is thus an optimal result for linear Lagrange elements.

Note that for this optimal convergence result more regularity is assumed as usually necessary.

Remark 6.26. Sometimes an a-priori error estimate in a weaker norm , as the L_2-norm, is from further interest. Usually such an estimate is proved by assuming $L_2 - H^2$ regularity, i.e.

$$\text{if } f \in L_2(\Omega) \text{ then } u \in H^2(\Omega) \wedge \|u\|_{2,\Omega} \preceq \|f\|_{L_2(\Omega)} \,, \tag{6.25}$$

and applying the *Aubin-Nitsche* trick. Note that due to Theorem 6.6 $p \in H^{\frac{1}{2}}(\Gamma_C)$. *Mortar* techniques also achieve optimal convergence in the L_2-norm. For the stabilised version of *nodal constraints* this isn't proved by the author, but the numerical results presented in Section 6.6 give rise to expect optimal convergence results also for the weaker L_2-norm.

6.4 2D Example

Assume $d = 2$ and $r = 1$ (linear ansatz functions). For this case an interpolation operator fulfilling (6.9a)- (6.9d) is constructed. Later on the uniform V_{0h}-ellipticity with respect to the norm $\|.\|_{V_{h,l}}$ is proved.

In the two dimensional case W_h is nothing else than the linear space which results from the ordinary linear and continuous ansatz for the new mesh given by joining the nodes of both sides ($\mathcal{N}_h^C := \mathcal{N}_{h,1}^C \cup \mathcal{N}_{h,2}^C$). Note that this new mesh in general degenerates and thus no interpolation operator with property (6.9a)- (6.9d) exists on W_h.

Nevertheless it's possible to define an interpolation operator $I_{W_h} : L_2(\Gamma_C) \to W_h$ such that

- $I^l w_h \quad = \quad w_h \qquad \forall\, v_h \in W_h$
- $\|I_{W_h} v\|_{0,E} \quad \preceq \quad \|v\|_{0,\omega_E} \quad \text{with}\, \omega_E = \text{int}\left(\bigcup \overline{K}|_{\Gamma_C} \,|\, \overline{K} \cap E \neq \emptyset,\, K \in \tau_h\right)$.

This is done by choosing for every $x_i \in \mathcal{N}_h^C$ the largest element σ_i and set

$$I_{W_h}(v)(x_i) := P_{\sigma_i}^{lin}(v)(x_i)\,.$$

$P_{\sigma_i}^{lin}$ is the projection operator defined in Section 5.2 for Scott-Zhang type operators and linear ansatz functions. Note that the largest element σ_i has measure of order $|\sigma_i| = O(h_l)$. That's the reason for the uniform boundness of $\|I_{W_h}\|_{L_2}$ in h_l. Define $I^l : L_2(\Gamma_C) \to \text{tr}\, V_{h,l}$

$$I^l u := I_N^l\, I_{W_h} u = \sum_{x_i \in \mathcal{N}_{h,l}^C} I_{W_h}(v)(x_i)\varphi_{i,l}\,,$$

where I_N^l is the ordinary nodal interpolation operator (see Section 5.2), $\varphi_{i,l}$ is the basis ansatz function φ_i restricted to Γ_C. This operator fulfils (6.9a)- (6.9d).

Lemma 6.27. *The operator $I^l = I_N^l I_{W_h}$ fulfils property* (6.9a)- (6.9d).

Proof. Because of Remark 6.9 it's possible to refer I^l as an operator $I^l : V_l \to V_{h,l}$. The proof is finished if it's proved that $I^l : V_l \to V_{h,l}$ is an Scott-Zhang operator and thus fulfils property (6.9b)-(6.9d). Following the original proof of Scott and Zhang [45] with a little modification, namely that σ_i on Γ_C is chosen as above , then the proof is done. Because the measure of σ_i is of order h_l ($|\sigma_i| \simeq h_l$) all steps of the proof of Scott and Zhang are valid. The only thing which has to be checked is the property (6.9a). Let $w_h \in W_h$ (or $w \in V_l$ such that $w|_{\Gamma_C} = w_h$), then

$$I^l(w_h)(x_i) = I_N^l I_{W_h}(w_h) = I_N^l(w_h) = w_h(x_i) \qquad \forall\, x_i \in \mathcal{N}_{h,l}^C\,.$$

□

The existence of this interpolation operator I^l guarantees the validity of the hole theory developed in Section 6.3. Especially the existence, uniqueness and convergence results are valid. For the stabilised mixed FEM the additional assumption, $a_{l,s}(.,.)$ is uniformly V_{0h}-elliptic with respect to the norm $\|.\|_{V_{h,l}}$, was made. For the 2D case this uniform ellipticity can also be proved.

Lemma 6.28. *Let $d = 2$ and assume a linear, quasi uniform Lagrange discretisation. Then the stabilised bilinear form $a_{l,s}(.,.)$ is uniformly V_{0h} elliptic, i.e.*

$$a_{l,s}(u_h, u_h) \succeq \|u_h\|^2_{V_{h,l}} \qquad \forall\, u_h \in \mathit{ker}\, B_{h,l-1}\,. \tag{6.26}$$

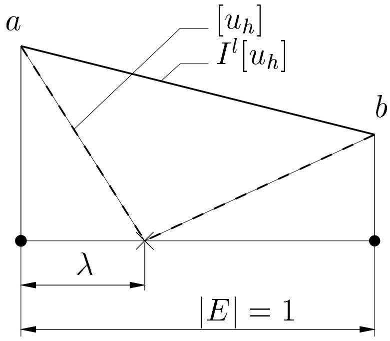
Proof. From Proposition 6.16 it's known that $a(.,.)$ is V_{0h}- elliptic, with respect to the norm $\|.\|_V$ and so it's enough to prove that

$$\sum_{E\in\mathcal{E}^C_{h,l}} h_E^{-2\alpha} \|\, [I^l u_h]\, \|^2_{0,E} \succeq \sum_{E\in\mathcal{E}^C_{h,l}} h_E^{-2\alpha} \|\, [u_h]\, \|^2_{0,E}\,.$$

Every element $u_h \in \ker B_{h,l-1}$ can be represented via

$$\begin{aligned} &\langle q_h, [v_h]\rangle_{Q'\times Q} = 0 && \forall\, q_h \in Q_{h,l-1} \\ \Leftrightarrow\quad & v_{h,1}(x_i) = v_{h,2}(x_i) && \forall\, x_i \in \mathcal{N}^C_{h,l-1}\,, \end{aligned}$$

and from the explicit representation of $W_h = \mathrm{tr}V_{h,1} + \mathrm{tr}V_{h,2}$ only the following extreme case has to be considered.



$\lambda \in (0,1)$

- • $x_i \in \mathcal{N}^C_{h,l}$ $\qquad a(1-x) + b$
- × $x_i \in \mathcal{N}^C_{h,l-1}$ $\qquad \begin{cases} a\,\dfrac{\lambda - x}{\lambda} & x \in [0,\lambda] \\ b\,\dfrac{x-\lambda}{1-\lambda} & x \in [\lambda, 1] \end{cases}$

$$\begin{aligned} \bullet \quad \int_E (I^l[u_h])^2\, dx &= \int_E (a(1-x) - b)^2\, dx = \tfrac{1}{3}(a^2 + b^2 + ab) \\ \times \quad \int_E [u_h]^2\, dx &= \int_{[0,\lambda]} a^2 \left(\frac{\lambda - x}{\lambda}\right)^2 dx + \int_{[\lambda,1]} b^2 \left(\frac{x-\lambda}{1-\lambda}\right)^2 dx \\ &= \tfrac{1}{3}(\lambda a^2 + (1-\lambda) b^2)\,. \end{aligned}$$

Summing up gives

$$2\int_E (I^l[u_h])^2\, dx - \int_E [u_h]^2\, dx = \frac{1}{3}\left((a^2(1-\lambda) + b^2\lambda) + (a+b)^2\right) \geq 0\,,$$

and thus the required result. □

With the existence of the interpolation operator and the uniform V_{0h}- ellipticity, with respect to the norm $\|.\|_{V_{h,l}}$ the optimal convergence results for *nodal constraints*, at least in 2D, is guaranteed.

6.5 Implementation of Stabilised FEM

For numerical purposes it's necessary to represent the discretised mixed equation (6.3), with $a_{l,s}(.,.)$ instead of $a(.,.)$, in matrix and vector notation. This is done in a standard way. The only interesting part is the explicit form of the stabilisation term. In this section it's not distinguished between $(u,p) \in V_h \times Q_{h,l}$ and the representing vector $(u,p) \in \mathbb{R}^{N_1+N_2} \times \mathbb{R}^{N_l^C}$. N_i is the number of free nodes in the domain Ω_i produced by the triangulation $\tau_{h,i}$ ($N := N_1 + N_2$). N_l^C denotes the number of free nodes at the boundary Γ_C produced from the triangulation of domain Ω_l. As in Section 6.3 the master surface is denoted by $l-1$ and the slave surface by l. Standard discretisation techniques give the following representation of the stabilisation term in $a_{l,s}(.,.)$ ($u \in V_h$):

$$\sum_{E \in \mathcal{E}_{h,l}^C} h_E^{-2\alpha} \int_E I^l[u]\, I^l[v]\, ds \;=\; \sum_{x_i, x_j \in \mathcal{N}_{h,l}^C} [u](x_i)[v](x_j) \underbrace{\sum_{E \in \mathcal{E}_{h,l}^C} h_E^{-2\alpha} \int_E \varphi_{i,l}\varphi_{j,l}\, ds}_{=:\,(C_{\alpha,l}^{-1})_{i,j}} .$$

With the setting

$$\begin{array}{lll} (B_l u)_i := [u](x_i) & & \forall\, x_i \in \mathcal{N}_{h,l}^C \\ A_{ij} \quad := a(\varphi_i, \varphi_j) & & \forall\, x_i,\, x_j \in \mathcal{N}_h \,, \end{array}$$

the stabilised bilinear form $a_{l,s}(.,.)$ can be representation by

$$a_{l,s}(u,v) \;=\; \langle Au, v\rangle_{l_2(\mathbb{R}^N)} + \langle B_l^T C_{\alpha,l}^{-1} B_l u, v\rangle_{l_2(\mathbb{R}^N)} .$$

Note that $A \in \mathbb{R}^{N \times N}$ and $B_l \in \mathbb{R}^{N \times N_l^c}$. Using the matrix vector representation of the bilinear form $b(.,.)$

$$b(q,v) \;=\; \sum_{x_i \in \mathcal{N}_{h,l-1}^C} q_i I^{l-1}[v](x_i) \;=\; \sum_{x_i \in \mathcal{N}_{h,l-1}^C} q_i [v](x_i) = \langle q, B_{l-1} v\rangle_{l_2(\mathbb{R}^{N_{l-1}^C})} ,$$

the discretised system (6.11) becomes the following indefinite system:

$$\begin{pmatrix} A + B_l C_{\alpha,l}^{-1} B_l & B_{l-1}^T \\ B_{l-1} & 0 \end{pmatrix} \begin{pmatrix} u \\ p_{l-1} \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} .$$

An algorithm to generate the matrix B_l is given in Section 7.6. From the discrete LBB-condition Proposition 6.23 it's known that the matrix B_l has full rank ($\operatorname{Rg} B_l = N_l^C$). Thus it's possible to rewrite above indefinite system into

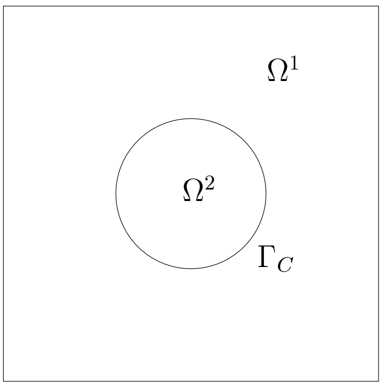
$$\begin{pmatrix} A & B_{l-1}^T & B_l^T \\ B_{l-1} & 0 & 0 \\ B_l & 0 & -C_{\alpha,l} \end{pmatrix} \begin{pmatrix} u \\ p_{l-1} \\ p_l \end{pmatrix} = \begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix} .$$

Note that the artificial introduce vector $p_l \in \mathbb{R}^{N_l^C}$ has no interpretation in the space $Q_{h,l}$! This substitution was only done for numerical purposes. It seems to be difficult to calculate the matrix $C_{\alpha,l}$. Assuming a uniform mesh, then calculating $C_{\alpha,l}$ is quit simple. Due to the uniformity of the mesh the stabilisation term $\sum\limits_{E \in \mathcal{E}_{h,l}^C} h_E^{-2\alpha} \int\limits_E I^l[u]\, I^l[v]\, ds$ is equivalent to $h_l^{-2\alpha} \int\limits_{\Gamma_C} I^l[u]\, I^l[v]\, ds$. Proceeding as above the stabilisation term is represented by $\langle B_l\, h_l^{-2\alpha} M_l^C\, B_l^T\, u, v\rangle_{l_2(\mathbb{R}^{N_l^C})}$. The matrix $(M_l^C)_{i,j} = \int\limits_{\Gamma_C} \varphi_{i,l}\varphi_{j,l}\, ds$ denotes the mass matrix at the boundary Γ_C produced by the triangulation of the domain Ω_l. For linear ansatz functions it's known that this mass matrix is spectrally equivalent to a diagonal matrix $(\overline{M}_l^C)_{ij} = \int\limits_{\Gamma_C} \varphi_{i,l}\, ds\, \delta_{ij}$, the so called lumped mass matrix. Thus $C_{\alpha,l}^{-1}$ can be replaced (without loosing any property like existence, uniqueness and convergence) by the diagonal matrix $\overline{C}_{\alpha,l}^{-1} = h_l^{-2\alpha}\overline{M}_l^C$, which is easy to invert. So it's enough to solve the system

$$\begin{pmatrix} A & B_{l-1}^T & B_l^T \\ B_{l-1} & 0 & 0 \\ B_l & 0 & -\overline{C}_{\alpha,l} \end{pmatrix} \begin{pmatrix} u \\ p_{l-1} \\ p_l \end{pmatrix} = \begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix}.$$

6.6 Numerical Example

To verify the results of Section 6.3 consider the following example and solve it numerically



$$\Delta u = -100\sin(10x)$$

$$u = \sin(10x) \qquad \text{on } \Gamma_D = \partial\Omega$$

Solution $u = \sin(10x)$

In Figure 6.3 the non-matching grid is visualised and in Figure 6.4 the solution for the mesh-parameter $h = \frac{1}{40}$ is visualised. The interesting graphics are the convergence rates in the H^1-norm (Figure 6.5) and the L_2 norm (Figure 6.6). For $\alpha = 1$, i.e. with stabilisation h^{-2} the optimal convergence rates are achieved, as it was proved in Section 6.3.

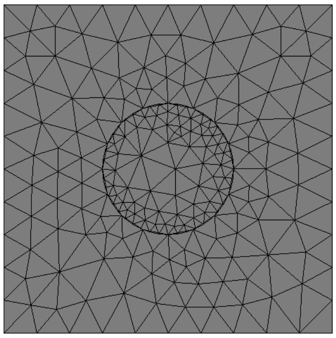


Figure 6.3: Mesh, $h = \frac{1}{10}$

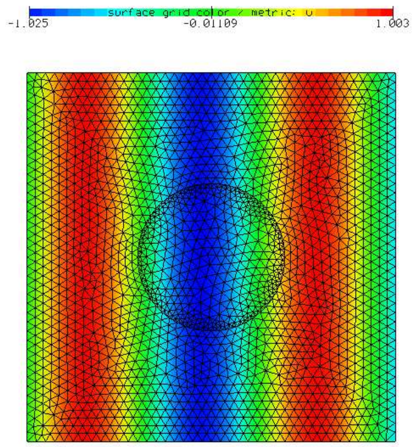


Figure 6.4: Solution, $h = \frac{1}{40}$

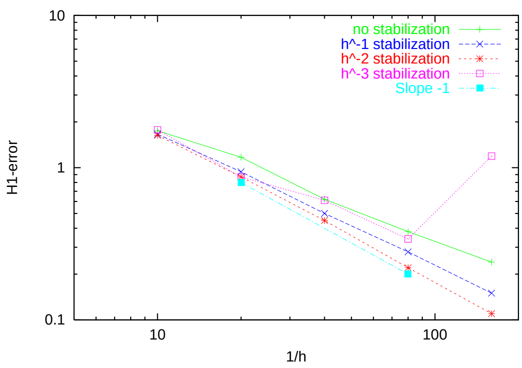


Figure 6.5: H^1-error

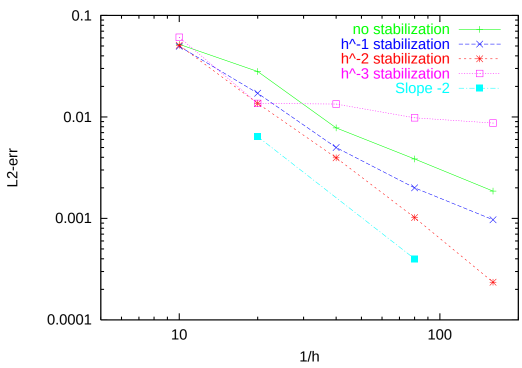


Figure 6.6: L_2-error

Chapter 7

The Body-Body Contact Problem

In Chapter 4 the *primal-* or *weak-* formulation of the body-body contact problem was presented. For this *primal-* or *weak-* formulation some equivalent formulations were presented. Also the unique solvability of the body-body contact problem was guaranteed in Section 4.4. The aim of this chapter is to reduce the body-body contact problem into something computable. This is done by standard FEM discretisation, which was presented in Chapter 5. The only thing which isn't obvious is how to approximate the convex sets K, N. There are several possibilities. One is to use *Mortar*. This approach is considered in BELGACEM, HILD AND LABORDE [6] [5] and shouldn't be considered in this thesis. Motivated from the results of Chapter 6 the body-body contact problem, i.e. the convex sets K, N, are approximated via *nodal constraints*. At least for the *Signorini* problem, this approach is very common and provides optimal results (see HASLINGER, HLAVÁČEK AND NEČAS [29], KIKUCHI AND ODEN [34], BREZZI, HAGER AND RAVIART [13] [14] and FALK [24]). The results achieved for the body-body contact problem with *nodal constraints* aren't optimal due to a similar reason as foe the non optimality of *nodal constraints* for non-matching grids.

Due to large scales of some body-body contact problems adaptive mesh refinement will be better suited to solve these problems. For variational inequalities the results on adaptive mesh refinement are very sparse (e.g. KORNHUBER [38], AINSWORTH, ODEN AND LEE [3]). To the knowledge of the author there is no a-posteriori error estimator which is well suited for the body-body contact problem and *nodal constraints*. For an augmented lagrangian algorithm CARSTENSEN, SCHERF AND WRIGGERS [15] presented an a-posteriori error estimator for the body-body contact problem, but this estimator doesn't fit into the solving algorithm used in this thesis. Thus an ad hoc refinement algorithm is presented which seems to give good results, at least for the 2D case. The refinement algorithm is based on standard error estimators for elliptic equalities, which can be found in VERFÜRTH [49] [50] or AINSWORTH AND ODEN [2].

In Section 7.1 a short review of the three important equivalent formulations of the body-body contact problem are given. These formulations are discretised by *nodal constraints* in Section 7.2. Some a-priori convergence results for this discretisation are presented in Section 7.3. The draw back for the *nodal constraints* is the missing convergence result, when no regularity of the solution is assumed. In Section 7.4 an ad-hoc adaptive mesh refinement algorithm is presented for the body-body contact problem with *nodal constraints* as discretisation. This refinement strategy seems to give "good" results. Motivated from Chapter 6 the body-body contact problem is stabilised in Section 7.5. The draw back for the stabilisation applied to the body-body contact problem is, that it isn't consistent anymore. Finally in Section 7.6 it's explained how to implement the constraints.

7.1 Formulations of the Body-Body Contact Problem

In Chapter 4, three equivalent formulations for the body-body contact problem (with $\langle A.,.\rangle_{V'\times V}$ V-elliptic) were presented. Because of the importance of these three equivalent formulations for this chapter, they are denoted again. The first one was the *primal-* or *weak-* formulation (4.4)

$$u = \operatorname*{argmin}_{v\in K} \frac{1}{2}\langle Av, v\rangle_{V'\times V} - \langle F, v\rangle_{V'\times V}\,, \tag{7.1}$$

with K the *set of admissible displacements* (4.2)

$$K = \left\{v \in V_0 \mid g - v_N^R \geq 0 \text{ a.e.}\right\}\,.$$

The second one was the *mixed or saddle-point* formulation

$$\begin{array}{lcll} \langle Au, v\rangle_{V'\times V} + \langle Bv, p\rangle_{Q'\times Q} & = & \langle F, v\rangle_{V'\times V} & \forall\, v \in V \\ \langle Bu, q\rangle_{Q'\times Q} & \leq & \langle G, q\rangle_{Q'\times Q} & \forall\, q \in N \\ \langle Bu - G, p\rangle_{Q'\times Q} & = & 0\,, & \end{array} \tag{7.2}$$

with $N := \mathcal{C}_+^0 = \left\{q \in H^{\frac{1}{2}}(\Gamma_C) \mid q \geq 0 \quad \text{a.e.}\right\}^0$.

Remark 7.1. For the *primal-* and for the *mixed-* formulation it's not necessary to assume V-ellipticity for A, ellipticity on K, which is equivalent to the ellipticity on $\ker B$, is enough to guarantee existence and uniqueness of the solution. Only for the third formulation, the *dual* formulation V-ellipticity is necessary.

In the numerical examples presented in this thesis (Chapter 9) $|\Gamma_D^1|$, $|\Gamma_D^2| > 0$ and with Theorem 4.27 A is V-elliptic. If one of the two bodies isn't fixed (w.l.o.g. $|\Gamma_D^2| = 0$), then it's possible to achieve a V-elliptic operator A by regularising A, i.e. adding some h-dependent mass term to A. This is some kind of Tikhonov regularisation and may work quite well. One problem is to find the correct h-dependent factor to keep the problem stable.

The *dual* formulation with A is V-elliptic reads as follows

$$p = \operatorname*{argmin}_{q\in N} \frac{1}{2}\langle BA^{-1}B^*q, q\rangle_{Q'\times Q} - \langle BA^{-1}F - G, q\rangle_{Q'\times Q}\,, \tag{7.3}$$

where u is calculated by solving the equality $Au = F - B^*p$.

Remark 7.2. A *dual* formulation exists also for a non V-elliptic operator A but the representation of this formulation isn't as simple as this one. The algorithm presented in the next chapter is essentially based on the *dual* formulation with V-elliptic operator A.

These three equivalent formulations enable a lot of different kinds of discretisation techniques and solving algorithms. In the next section the *primal* and the *mixed* formulation will be discretised and it turns out that both discretisation are equivalent.

7.2 Discretisation of the Body-Body Contact Problem

For discretising the body-body contact problem the FEM, presented in Chapter 5, is applied. Thus the discretisation of the space V is done in a standard way, i.e.

$$V_h = \{ v_h \in C_{0,D}(\overline{\Omega}) \,|\, v_h|_T \in \mathcal{P}^r(T) \quad \forall T \in \tau_h \} .$$

It's also possible to use FEM to discretise Q, keeping in mind that the pair $V_h \times Q_h$ has to be stable. This will result into *Mortar* techniques, which shouldn't be discussed in this thesis. For discretisation of the body-body contact problem with *Mortar* techniques refer to Belgacem, Hild and Laborde [6] [5]. In this thesis the discretisation of the space Q should be done by *nodal evaluating linear functionals*, i.e.

$$Q_{h,l} = \left\{ q \in Q \;\mid\; q(v) = \sum_{x_i \in \mathcal{N}^C_{h,l}} q_i \, (I^l v)(x_i) \quad q_i \in \mathbb{R} \right\} .$$

This *nodal evaluating linear functionals* and their properties are known from Chapter 6.

To discretise formulations presented above it's not enough to discretise the spaces V, Q, it's necessary to discretise the convex sets K, N. The first idea to approximate the convex sets K, N may be to use $K_h = K \cap V_h$, $N_h = N \cap Q_h$. This sets would have a lot of advantages concerning a-priori and a-posteriori error estimates, but the disadvantage is that these sets can't be handled numerically, at least not with a reasonable effort. Further more the set $N \cap Q_{h,l}$ may be empty if I^l doesn't preserve positive functions. A very common approximation of the convex set K is the setting

$$K_{h,l} = \{ v_h \in V_h \,|\, {v_h}^R_N(x_i) \leq g(X_i) \quad \forall x_i \in \mathcal{N}^C_{h,l} \} . \tag{7.4}$$

This approximation $K_{h,l}$ of the convex set K is very simple to handle numerically, because the admissibility of $v_h \in V_h$ is tested by evaluating v_h at certain nodes in $\mathcal{N}_h$. The big disadvantage of this approximation is that $K_{h,l}$ is no subset of K anymore. To prove that the solution of the discretised *primal* formulation u_h converges strongly to the solution of the body-body contact problem u assumptions (5.15a)-(5.15b) have to be fulfilled (Theorem 5.9). The author isn't in the position to prove (5.15a) and up to the knowledge of the author there is no proof.

The discretised *primal* formulation is also equivalent to a *mixed* and a *dual* formulation. Both are achieved by introducing artificial *Lagrange* multipliers. This isn't very satisfying because this artificial *Lagrange* multipliers have no interpretation in the space Q.

To get rid of this artificial *Lagrange* multipliers it's necessary to discretise the *mixed* or *saddle point* formulation. The discrete spaces V_h, $Q_{h,l}$ are still fixed. The only thing which is missing is the approximation of the set N. In Chapter 6 it was seen that I_Q^l is an interpolation operator from Q into $Q_{h,l}$. To approximate the set N consider $I_Q^l q \qquad \forall\, q \in N$ (equation (6.12)) explicitely.

$$I_Q^l q \;=\; \sum_{x_i \in \mathcal{N}_{h,l}^C} \langle \varphi_{i,l}, q\rangle_{Q' \times Q} I^l(.)(x_i) \tag{7.5}$$

If the ansatz functions $\varphi_{i,l}$ have polynomial degree less than three, then they are positive and because $q \in N$ it's deduced that $\langle \varphi_{i,l}, q\rangle_{Q'\times Q} \;\geq\; 0$. Thus approximate N via

$$N_{h,l} \;=\; I_Q^l N \;=\; \left\{ q_h \in Q_{h,l} \,|\, q_h = \sum_{x_i \in \mathcal{N}_{h,l}^C} q^i I^l(.)(x_i) \quad \forall\, q_i \geq 0 \right\}. \tag{7.6}$$

Also for this approximation $N_{h,l}$ is no subset of N.

Remark 7.3. If the interpolation operator $I^l(.)$ is constructed like in Chapter 6 then $I^l(.)$ doesn't map positive functions into positive ones and thus no element of $N_{h,l}/\{0\}$ is element of N.

The solution u_h of the discretised mixed system fulfils

$$\langle Bu_h - G, q_h - p_h\rangle_{Q'\times Q} \leq 0 \qquad \forall\, q_h \in N_{h,l}$$

and if $I^l(g) \;=\; g$ this is nothing else than $(Bu_h \;-\; g)(x_i) \;\leq\; 0 \quad \forall\, x_i \;\in\; \mathcal{N}_{h,l}^C$. In fact the convex set $K_{h,l}$ can be represented via

$$K_{h,l} \;=\; \{\, v_h \in V_h \,|\, \langle Bv_h - G, q_h\rangle_{Q'\times Q} \;\leq\; 0 \qquad \forall\, q_h \in N_{h,l} \,\}\,.$$

Thus the discretised *primal* formulation and the discretised *mixed* formulation and also the discretised *dual* formulation are equivalent. Especially the artificial introduced *Lagrange* multipliers for the discretised *primal* formulation becomes a meaning in the space Q.

7.3 Convergence Results

To achieve convergence results the existence and uniqueness of the discrete problem has to be guaranteed. In Chapter 5 it was proved that it's enough to check the uniform kernel ellipticity and the uniform LBB condition. Because it was assumed that A is V-elliptic, the kernel ellipticity is trivial. The LBB condition is in general more complicated, but in the special case that $X^{(l)}$ is the identity map and n is a constant unit vector, the discrete uniform LBB condition is trivial. The proof is similar to that one of Proposition 6.15. The only thing what has to be proved is that I^l is a *Fortin* operator on the space $V_l \times \{0\}$, which is trivial.

Remark 7.4. The numeric examples presented at the end of this thesis fulfil both, that A is V-elliptic, because $|\Gamma_D^1|$, $|\Gamma_D^2| > 0$ and the normal unit vector n is constant. Note that n is constant is nothing else than a special choice of the parametrisations $X^{(j)}$.

In the section before it was mentioned that the author isn't in the position to prove condition (5.15a) for the *primal formulation* and condition (5.15b) for the *dual formulation* and so Theorem 5.9 can't be applied, i.e. it can't be proved that $\lim_{h\to 0^+} \|u_h - u\|_V = 0$ for the *primal* formulation and $\lim_{h\to 0^+} \|p_h - p\|_Q = 0$ for the *dual* formulation, without assuming any regularity. Assuming more regularity for the solution u of the *primal* formulation the following proposition holds.

Proposition 7.5. *Let* $g \in H^{\frac{d-1}{2}+\epsilon}(\Gamma_C)$, $u \in (H^{\frac{d}{2}+\epsilon}(\Omega_1) \times H^{\frac{d}{2}+\epsilon}(\Omega_2)) \cap K$, *be solution of* (7.1)*, where* ϵ *is an arbitrary small number,* $u_h \in K_{h,l}$ *be the corresponding solution of the discretised formulation* (7.1)*. Furthermore let the polynomial degree of the ansatz functions be less than three, then*

$$\lim_{h\to 0^+} \|u - u_h\|_V = 0 \,. \tag{7.7}$$

Proof. The prove is done by using Lemma 5.8, i.e. by estimating the right hand side of the following inequality.

$$\begin{aligned} \langle Ae, e\rangle_{V'\times V} \;\le\;& \langle F, u - v_h\rangle_{V'\times V} + \langle F, u_h - v\rangle_{V'\times V} \\ & + \langle Ae, u - v_h\rangle_{V'\times V} + \langle Au, v - u_h\rangle_{V'\times V} + \langle Au, v_h - u\rangle_{V'\times V} \,, \end{aligned}$$

which is valid $\forall\, v \in K \,\forall\, v_h \in K_{h,l}$. Before estimating the last inequality, i.e. choosing v and v_h, some facts are denoted.

- Because of the regularity of u and g and the embedding theorem Theorem 3.4 it's known that u and g are continuous and thus the nodal interpolation operator I_N is well defined. Furthermore it's known that $Bu - g \le 0$ a.e. and due to the continuity of u and g this inequality is valid everywhere. Thus the function $I_N u$ fulfils $(BI_N u - g)(x_i) \le 0 \quad \forall\, x_i \in \mathcal{N}_{h,l}^C$. This is nothing else than $I_N u \in K_{h,l}$.

- Condition (5.15b) is valid. For the proof assume $w_h \in K_{h,l}$ with $w_h \rightharpoonup w$. To prove that $w \in K$ consider $\langle Bw - G, q\rangle_{Q'\times Q}$ for all $q \in N$.

  $$\begin{aligned} \langle Bw_h - G, q\rangle_{Q'\times Q} &= \langle Bw_h - G, q - I_Q^l q\rangle_{Q'\times Q} + \langle Bw_h - G, I_Q^l q\rangle_{Q'\times Q} \\ &\overset{(*)}{\le} \langle Bw_h - G, q - I_Q^l\rangle_{Q'\times Q} \le \|Bw_h - G\|_{Q'} \|q - I_Q^l q\|_Q \end{aligned}$$

  $(*)$ $I_Q^l q \in N_{h,l} \quad \forall\, q \in N$ because the ansatz functions $\varphi_{i,l}$ have polynomial degree less than three.

  Using the weak convergence of $w_h \rightharpoonup w$, which also guarantees that w_h is bounded (uniform boundness principle see Heuser [31, Satz 40.2]) and Lemma 6.13 it's proved that $\langle Bw - G, q\rangle_{Q'\times Q} \le 0 \quad \forall\, q \in N$ and thus $w \in K$.

- u_h is uniformly bounded due to Theorem 4.11 and thus has a weak converging subsequence (*Eberlein-Shmulyan* see HEUSER [31, Satz 60.6]). W.l.o.g. assume that $u_h \rightharpoonup u^*$. Note that $u^* \in K$ because (5.15b) is valid.

Now consider above inequality an set $v_h = I_N u$ and $v = u^*$. Noting that A is V-elliptic, $I_N u$ converges strong to u the assertion is proved for every subsequence and thus for the hole sequence. □

The last proposition guarantees convergence under some regularity assumptions. Assuming more regularity a convergence rate is expected. The next proposition provides a convergence rate which is similar to that one of Chapter 6 for *nodal constraints* without any stabilising term.

Proposition 7.6. *Assume for B and the space $V_h \times Q_{h,l}$ the uniform LBB condition. Let $g \in H^{\frac{d-1}{2}+\epsilon}(\Gamma_C)$, $(u,p) \in ((H^2(\Omega_1) \times H^2(\Omega_2)) \cap V) \times (L^2(\Gamma_C) \cap N)$, be the unique solution of (7.2) and (u_h, p_h) the corresponding unique solution of the discretised formulation. Furthermore assume that the parametrisations $X^{(i)}$ are such that $Bu \in H^{\frac{d-1}{2}+\epsilon}(\Gamma_C)$. Then the following convergence rate holds.*

$$\|u - u_h\|_V + \|p - p_h\|_Q \le O(h^{\frac{1}{2}}) \tag{7.8}$$

Proof. First prove the assertion for the primal variable u by using inequality (5.16) from Lemma 5.6. I.e. $\forall\, v_h \in K_{h,l} \,\forall\, q_h \in N_{h,l}$ the following inequality is valid.

$$\begin{aligned}\langle Ae, e\rangle_{V'\times V} \;\le\; & \langle Ae, u - v_h\rangle_{V'\times V} + \langle B(u_h - v_h), p - q_h\rangle_{Q'\times Q} \\ & + \langle B(u - v_h), q_h\rangle_{Q'\times Q} + \langle Bu - G, p - q_h\rangle_{Q'\times Q}\end{aligned}$$

Like in the proof of Proposition 7.5 it follows that $I_N u \in K_{h,l}$ and $I_Q^l p \in N_{h,l}$ and thus set $v_h = I_N u$ and $q_h = I_Q^l p$. For notation set $X = L_2(\Gamma_C)$ and $V^+ = H^2(\Omega_1) \times H^2(\Omega_2)$.

- $\langle Ae, u - I_N u\rangle_{V'\times V} \overset{C.S.}{\le} \alpha\|e\|_A^2 + \frac{1}{\alpha}\|u - I_N u\|_A^2 \overset{(5.8)}{\preceq} \alpha\|e\|_A^2 + \frac{h^2}{\alpha}\|u\|_{V^+}^2$

- $$\begin{aligned}&\langle B(u_h - I_N u), p - I_Q^l p\rangle_{Q'\times Q} \\ &\quad= \langle B(u_h - u), p - I_Q^l p\rangle_{Q'\times Q} + \langle B(u - I_N u), p - I_Q^l p\rangle_{Q'\times Q} \\ &\quad\overset{C.S.}{\le} \|B\|_{L(V,Q')}\left(\|e\|_V\|p - I_Q^l p\|_{V^+}\|u - I_N u\|_V\|p - I_Q^l p\|_Q\right) \\ &\quad\overset{C.S.+(4.22)}{\le} \beta\|e\|_A^2 + \frac{1}{\beta}(\alpha_1\|B\|_{L(V,Q')})^2\|p - I_Q^l p\|_Q^2 \\ &\qquad + \frac{\|B\|_{L(V,Q')}}{2}(\|u - I_N u\|_V^2 + \|p - I_Q^l p\|_Q^2) \\ &\quad\overset{\text{Lemma 6.13}+(5.8)}{\preceq} \beta\|e\|_A^2 + \frac{h_l}{\beta}\|p\|_X^2 + h^2\|u\|_{V^+}^2\end{aligned}$$

- $$\langle B(u - I_N u), I_Q^l p\rangle_{Q'\times Q} \overset{C.S.}{\leq} \|B\|_{L(V,Q')}\|u - I_N u\|_V \|I_Q^l p\|_Q \overset{\text{Lemma 6.13+(5.8)}}{\preceq} h\|u\|_{V^+}\|p\|_Q$$

- Set $Y' = H^1(\Gamma_C)$ and Y its dual space, then

$$\langle Bu - G, p - I_Q^l p\rangle_{Q'\times Q} \leq \|Bu - G\|_{Y'}\|p - I_Q^l p\|_Y \overset{\text{Lemma 6.13}}{\preceq} h_l\|Bu - G\|_{Y'}\|p\|_X .$$

Choosing α, β small enough (independent of h) then

$$\|u - u_h\|_V^2 \overset{(4.22)}{\preceq} \|u - u_h\|_A^2 \leq h\left(\|u\|_{V^+}^2 + \|p\|_X^2 + \|Bu - G\|_{Y'}\|p\|_X + \|p\|_Q\|u\|_{V^+}\right) .$$

The missing part is the convergence of the *dual* part $\|p - p_h\|_Q \leq \|p - q_h\|_Q + \|q_h - p_h\|_Q$.

$$\begin{aligned}\|q_h - p_h\|_Q &\overset{(5.13)}{\preceq} \sup_{v_h\in V_h} \frac{\langle Bv_h, q_h - p_h\rangle_{Q'\times Q}}{\|v_h\|_V}\\ &\overset{(7.2)}{=} \sup_{v_h\in V_h} \frac{\langle A(u_h - u), v_h\rangle_{V'\times V} - \langle Bv_h, q_h - p\rangle_{Q'\times Q}}{\|v_h\|_V}\\ &\overset{(4.22)}{\leq} \|u - u_h\|_V + \|B\|_{L(V,Q')}\|p - q_h\|_Q\end{aligned}$$

With $q_h = I_Q^l p$ the assertion is proved. □

Remark 7.7. The convergence result $\|u-u_h\|_V+\|p-p_h\|_Q \leq O(h^{\frac{1}{2}})$ is the best possible for the assumed regularity, but for the assumed regularity in the *primal variable* the regularity in the *dual variable* is usually better ($p \in H^{\frac{1}{2}}(\Gamma_C)$). Then this result isn't the best possible. Comparing the result with that one achieved in Chapter 6 for *nodal constraints* without stabilisation, it's the best result which can be expected with the theory given in this thesis.

Remark 7.8. Some regularity results for the body-body contact problem can be found in Boieri, Gastaldi and Kinderlehrer [7].

7.4 Ad-Hoc Adaptive Mesh Refinement

One of the numeric example (the real life problem) has a very large scale. Using a uniform mesh the number of unknowns will become too large before the contact area is "correctly" described. This problem can be solved by generating the mesh by hand or by using adaptive mesh refinement. Up to the knowledge of the author there are no efficient element error estimators for the body-body contact problem with *nodal constraints* as discretisation. Thus it's necessary to use ad-hoc refinement algorithm. In Ainsworth and Oden [2] some gradient based error estimators are presented which can be used.

The ad-hoc adaptive mesh refinement which is used in this thesis is based on inequality (5.17b) from Lemma 5.7.

$$\begin{aligned}\langle Ae, e\rangle_{V'\times V} &= \{\langle F, e-e_h\rangle_{V'\times V} - \langle Au_h, e-e_h\rangle_{V'\times V} - \langle B(e-e_h), p_h\rangle_{Q'\times Q}\}\\ &\quad + \{\langle Bu - G, p_h\rangle_{Q'\times Q}\} + \{\langle Bu_h - G, p\rangle_{Q'\times Q}\}\end{aligned}$$

There are three terms which should be considered separately.

-

$$\begin{aligned}&\langle Au_h - F, e-e_h\rangle_{V'\times V}\\ &= \sum_{T\in\tau_h} \langle \sigma(u_h), \operatorname{grad}(e-e_h)\rangle_{0,T} - \langle \hat{F}, e-e_h\rangle_{0,T} - \langle L, e-e_h\rangle_{0,\partial T\cap\Gamma_N}\\ &\overset{(3.15)}{=} -\sum_{T\in\tau_h} \langle \operatorname{div}\sigma(u_h) + \hat{F}, e-e_h\rangle_{0,T} + \langle \sigma^{(\nu)}, e-e_h\rangle_{0,\partial T} - \langle L, e-e_h\rangle_{0,\partial T\cap\Gamma_N}\\ &= -\sum_{T\in\tau_h} \langle \operatorname{div}\sigma(u_h) + \hat{F}, e-e_h\rangle_{0,T} + \sum_{E\in\mathcal{E}_h^\Omega} \langle [\sigma^\nu(u_h)], e-e_h\rangle_{0,E}\\ &\quad + \sum_{E\in\mathcal{E}_h^N} \langle \sigma^{(\nu)} - L, e-e_h\rangle_{0,E} + \sum_{E\in\mathcal{E}_{h,l}^C\cup\mathcal{E}_{h,l-1}^C} \langle \sigma^{(\nu)}, e-e_h\rangle_{0,E}\end{aligned}$$

$[\,.\,]$ is defined as follows: Let $E\in\mathcal{E}_h^\Omega$, then two unique elements $T, K\in\tau_h$ exists with $E = T\cap K$. With this $[\sigma^{(\nu)}(u)] := \sigma^{(\nu)}(u)_T - \sigma^{(\nu)}(u)_K$. Note that ν is a unit normal vector on E. The error term

$$\sum_{E\in\mathcal{E}_{h,l}^C\cup\mathcal{E}_{h,l-1}^C} \langle \sigma^{(\nu)}, e-e_h\rangle_{0,E}\,,$$

together with the error term $\langle B(e-e_h), p_h\rangle_{Q'\times Q}$ can't be handled by the author and are thus omitted. The element error $\eta_{T,1}$ can thus be defined by

$$\begin{aligned}\eta_{T,1}^2 &:= h_T^2\|\operatorname{div}\sigma(u_h) + f\|_{0,T}^2 + \frac{1}{2}\sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_h^\Omega} h_T\|[\sigma^{(\nu)}(u_h)]\|_{0,E}^2\\ &\quad + \sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_h^N} h_T\|\sigma^{(\nu)} - L\|_{0,E}^2\,.\end{aligned}$$

Using only this element error $\eta_{T,1}$ for generating an adaptive mesh the mesh will look like that one illustrated in Figure 7.3. It can be seen that the mesh becomes more finer at the slave surface $l-1$ than at the master surface l. The hanging nodes at the slave surface have no reasonable information for the contact. Due to the more curved surface of the slave surface the gradient of the primal variable u becomes larger. Especially gradient based error estimators recognise this and refine the slave surface again. The mesh becomes worser and worser. The idea to prevent this may be to penalise the penetration of the slave surface. This can be done by considering this penetration in the operator A, similar to Section 6.3 or by refining elements at the master surface, which are penetrated. The first idea is presented in Section 7.5 and the second one is done now, by considering further terms of (5.17b).

- The error term which describes that $u_h \notin K_{h,l}$ is $\langle Bu_h - G, p\rangle_{Q'\times Q}$. This term can also be considered as the term which estimates the error resulting from hanging nodes. In the hope that a correct estimate of this term produces a better mesh than that one which is produced by using only $\eta_{T,1}$, this term is considered in the following. Let $w_h \in V_h$ such that $\langle B(u_h - w_h) - G, q\rangle_{Q'\times Q}, \leq 0 \quad \forall\, q \in N$. Note that this (if the parametrisation is smooth enough) is equivalent to $B(u_h - w_h) - G \leq 0$ on Γ_C and because u_h, $w_h \in V_h$ this is equivalent to $B(u_h - w_h)(x_i) \leq g(x_i) \quad \forall\, x_i \in \mathcal{N}^C_{h,l} \cup \mathcal{N}^C_{h,l-1}$. The term $\langle Bu_h - G, P\rangle_{Q'\times Q}$ can now be estimated by

$$\begin{aligned}\langle Bu_h - G, p\rangle_{Q'\times Q} &= \langle B(u_h - (u_h - w_h)), p\rangle_{Q'\times Q} + \langle B(u_h - w_h) - G, p\rangle_{Q'\times Q}\\ &\leq \langle Bw_h, p\rangle_{Q'\times Q}\\ &= \langle \max\{0, Bw_h\}, p\rangle_{Q'\times Q} - \langle \max\{0, -Bw_h\}, p\rangle_{Q'\times Q}\\ &\leq \langle \max\{0, Bw_h\}, p\rangle_{Q'\times Q} \leq \|p\|_Q \|\max\{0, Bw_h\}\|_{Q'}\,.\end{aligned}$$

To get an idea how to chose w_h consider Figure 7.1.

There are several possibilities to choose w_h. Consider only the two cases that either

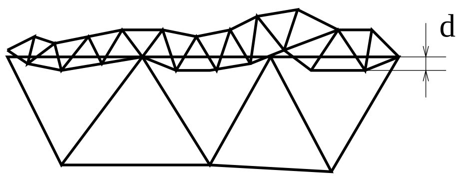


Figure 7.1: sketch of the penetration (hanging nodes)

the master surface l or the slave surface $l-1$ is displaced such that $B(u_h - w_h) \leq G$. I.e.

$$w_h = \sum_{x_i \in \mathcal{N}^C_{h,j}} d_i \varphi_i\,. \tag{7.9}$$

j is either l or $l-1$. The meaning of d_i is illustrated in Figure 7.1 and can be calculated by a post processing step. To calculate d_i fix the node x_i and take all elements $T \in \tau_{h,j}$ such that $x_i \in T$. For this elements calculate the maximal value of penetration. This maximal value is d_i. This calculation is simple to implement and is of optimal time complexity.

$$\begin{aligned}\langle Bu_h - G, p\rangle_{Q'\times Q} &\preceq \|p\|_Q \sqrt{\sum_{T\in\tau_{h,j}} \|\sum_{x_i\in\mathcal{N}^C_{h,j}} d_i\varphi_i\|^2_{1,\omega(T)}}\\ &\overset{\text{Theorem 5.2}}{\preceq} \|p\|_Q \sqrt{\sum_{T\in\tau_{h,l}} h_T^{-2} \|\sum_{x_i\in\mathcal{N}_{h,j}} d_i\varphi_i\|^2_{0,\omega(T)}}\\ &\overset{(*)}{\preceq} \|p\|_Q \sqrt{\sum_{T\in\tau_{h,j}} d_T^2 h_T^{d-2}}\end{aligned}$$

(*) It's well known that the mass matrix M_j is spectral equivalent to $h_j^d I$. The same is true for the local part

$$\| \sum_{x_i \in \mathcal{N}_{h,j}^C} d_i \varphi_i \|_{0,\omega(T)} = \sum_{x_i, x_n \in \mathcal{N}_{h,j}^C} d_i d_n \langle \varphi_i, \varphi_n \rangle_{0,\omega(T)} \simeq \sum_{\substack{x_i \in \mathcal{N}_{h,j}^C \\ x_i \in \omega(T)}} d_i^2 h_T^d .$$

The value d_T^2 is nothing else than

$$d_T^2 := \sum_{\substack{x_i \in \mathcal{N}_{h,j}^C \\ x_i \in \omega(T)}} d_i^2 .$$

Note that φ_i isn't the ansatz function restricted to Γ_C.

The last estimate provides a local error estimate which can be added with a certain constant (depending on $\|p\|Q$ to $\eta_{T,1}$. Define this additional term as

$$\eta_{T,2}^2 = d_T^2 h_T^{d-2} .$$

From the construction it's possible that $\eta_{T,2}$ is an additional error added to the slave surface $l-1$ or to the master surface l. In Figure 7.3 it was illustrated that the slave surface becomes a lot of hanging nodes if $\eta_{T,1}$ is used. Thus the additional term $\eta_{T,2}$ is added to the master surface l (for penalising penetration or hanging nodes). In Figure 7.4 the mesh generated by using the element error $\eta_{T,1} + \eta_{T,2}$ ($\eta_{T,2}$ at the master surface l) is illustrated. The mesh has much less hanging nodes and seems to be very satisfying

- The last term remaining is $\langle Bu - G, p_h \rangle_{Q' \times Q} \leq \inf_{v_h \in K_{h,l}} \langle B(u - v_h), p_h \rangle_{Q' \times Q}$. It is also very difficult to handle this term and thus it's neglected.

For adaptive mesh refinement set η_T either $\eta_{T,1}$ or $\eta_{T,1} + \eta_{T,2}$. For η_T it's expected that

$$\sum_{T \in \tau_h} \eta_T^2 \preceq \mathrm{TOL} .$$

TOL is the tolerance and chosen by the user. To minimise the number of degrees of freedom during the refinement it's expected that the error η_T^2 is equally distributed. Thus an element $T \in \tau_h$ is refined if

$$\eta_T^2 \leq \frac{\mathrm{TOL}}{|\tau_h|} .$$

The missing and unknown constant is absorbed with TOL. $N_T = |\tau_h|$ is the number of elements.

Remark 7.9. For Figure 7.3-Figure 7.5 the lower body is that one with the master surface l and the upper one that one with the slave surface.

7.5 Stabilisation of the Body-Body Contact Problem

In Chapter 6 a consistent stabilisation term was added to the operator A such that it was possible to achieve better convergence results for the *primal variable*. This may be a motivation to do the same for the body-body contact problem. In Section 7.4 it was mentioned that the adaptive mesh generator based on the element error $\eta_{T,1}$ isn't very satisfying due to the fine mesh at the slave surface. The guessed reason for this fine mesh were the hanging nodes which result into a curved surface and thus into large gradients for the discretised solution u_h. One idea to prevent this was to penalise hanging nodes. For a moment assume that the slave surface is the surface l. A consistent penalisation of the hanging nodes, motivated by Section 6.3 will be

$$\langle A_{l,s}.,.\rangle_{V'\times V} \;=\; \langle A.,.\rangle_{V'\times V} + \sum_{E\in\mathcal{E}^C_{h,l}} h_E^{-2\alpha}\langle I^l(B\,. - G)_+, I^l(B\,. - G)_+\rangle^2_{0,E}\,,$$

where $(\,.\,)_+ := \max\{0,\,.\,\}$ represents the positive part of a function. It's obvious that the stabilised version is consistent, because the *admissible displacements* $v \in K$ fulfil $Bv - G \leq 0$ a.e.. The term $(B\,. - G)_+$ represents the penetration and thus penalises functions which aren't in the set of *admissible displacements*. Consider an element $v_h \in K_{h,l-1}$. Then the hanging nodes (see Figure 7.1), which penetrate the body $l-1$ are penalised. Thus it's expected that the curvature of the slave surface becomes smoother and the adaptive error estimator doesn't refine the slave surface as strong as in the original formulation. The stabilisation term introduced above can't be handled numerically with a reasonable effort, thus the stabilisation term is changed into

$$\langle A_{l,s}.,.\rangle_{V'\times V} \;=\; \langle A.,.\rangle_{V'\times V} + \sum_{E\in\mathcal{E}^C_{h,l}} h_E^{-2\alpha}\langle \big(I^l(B\,. - G)\big)_+, \big(I^l(B\,. - G)\big)_+\rangle^2_{0,E}\,.$$

In general this stabilisation term isn't consistent because the interpolation operator I^l doesn't preserve positive functions. Thus it isn't guaranteed that the stabilised version doesn't change the problem seriously. The last stabilised version can be handled numerically, but also in this case the effort for it is too much. Having in mind only the idea of penalising hanging nodes which penetrate, the stabilisation can be changed into

$$\langle A_{l,s}.,.\rangle_{V'\times V} \;=\; \langle A.,.\rangle_{V'\times V} + \sum_{E\in\mathcal{E}^C_{h,l}} h_E^{-2\alpha}\langle I^l_+(B\,. - G), I^l_+(B\,. - G)\rangle^2_{0,E}\,, \tag{7.10}$$

with $I^l_+(.) := \sum_{x_i\in\mathcal{N}^C_{h,l}} \max\{0, I^(.)(x_i)\}\varphi_{i,l}$. Note that $I^l_+(I^l(\,.\,)) \;=\; I^l_+(\,.\,)$. The last stabilisation term is simple to handle, especially for a uniform mesh and linear *Lagrange* elements. The reason is similar to that of Section 6.5. Writing down the stabilisation term explicitely gives

$$h^{-2\alpha} \sum_{x_i,x_j\in\mathcal{N}^C_{h,l}} I^l_+(B\,. - G)(x_i) I^l_+(B\,. - G)(x_j) \underbrace{\langle\varphi_{i,l},\varphi_{j,l}\rangle_{L_2(\Gamma_C)\times L_2(\Gamma_C)}}_{=\;(M^C_l)_{ij}}\,.$$

It's well known that for linear ansatz functions the mass matrix M_l^C is spectral equivalent to the lumped mass matrix $\overline{M}_l^C$. Thus the stabilisation term is equivalent to

$$h^{-2\alpha} \sum_{x_i \in \mathcal{N}_{h,l}^C} (I_+^l (B. - G)(x_i))^2 (\overline{M}_l^C)_{ii} \,.$$

Introducing artificial *Lagrange* multipliers on the space $\mathbb{R}_+^{N_l^C}$ the stabilisation term can be written as supremum of a quadratic constraint minimisation problem.

$$\sum_{x_i \in \mathcal{N}_{h,l}^C} (I_+^l (B. - G)(x_i))^2 (\overline{M}_l^C)_{ii} = - \inf_{q \in \mathbb{R}_+^{N_l^C}} \langle \overline{M}_l^C q, q \rangle_{l_2(\mathbb{R}^{N_l^C})} - 2 \sum_{x_i \in \mathcal{N}_{h,l}^C} q_i I^l (B. - G)(x_i)$$

Thus the stabilised problem can be denoted as saddle point problem or equivalent as mixed inequality system.

$$\begin{array}{llll} \langle Au, v \rangle_{V' \times V} + \langle B_{l-1} v, p_{l-1} \rangle_{Q' \times Q} + \langle I^l B_l v, p_l \rangle_{Q' \times Q} & = & \langle F, v \rangle_{V' \times V} & \forall\, v \in V \\ \langle B_{l-1} u - G, q_{l-1} - p_{l-1} \rangle_{Q' \times Q} & \leq & 0 & \forall\, q_{l-1} \in N_{l-1} \\ \langle I^l \left(B_l u - G \right), q_l - p_l \rangle_{Q' \times Q} - \langle (\overline{M}_l^C)^{-1} p_l, q_l - p_l \rangle_{Q' \times Q} & \leq & 0 & \forall\, q_l \in N_l \end{array} \tag{7.11}$$

Note that in this case the artificial *Lagrange* multipliers have no interpretation in Q. The identification of the vector $q_l \in \mathbb{R}_+^{N_l^C}$ with the functional $q_l \in N_l$ is only an artificial one and possible because the inverse of the lumped mass matrix $(\overline{M}_l^C)^{-1}$ is interpreted as an operator mapping $Q_{h,l} \to \operatorname{tr} V_{h,l}$. More precisely

$$(\overline{M}_l^C)^{-1} := \sum_{x_i \in \mathcal{N}_{h,l}^C} \varphi_{i,l} (\overline{M}_l^C)_{ii}^{-1} \varphi_{i,l}^* \,,$$

with $\varphi_{i,l}^* : Q_{h,l} \to \mathbb{R} \quad q_h \mapsto \langle \varphi_{i,l}, q_h \rangle_{Q' \times Q}$.

This system fits into the theory presented in the last chapters and also fits into the class of problems which can be solved by the numerical algorithm presented in the next chapter.

Remark 7.10. Numerical calculations with the stabilised version are inefficient because of missing robust preconditioners for parameter dependent problems. Nevertheless calculations with this stabilised versions were done. The only result which should be presented here is the mesh generated by the solution of the stabilised version using only $\eta_{T,1}$ for the adaptive mesh refinement (see Figure 7.5). The mesh generated by this version seems to be the best one compared with the other two possibilities. To get an idea how to construct a robust preconditoner for the stabilised version and thus how to solve the stabilised version efficiently, refer to SCHÖBERL [44].

7.6 Generation of B_l and G_l

In the previous section the discretisation of the body-body contact problem was introduced. The constraints for the discretised problem were reduced to some *nodal constraints*. The link between the values at the node $x_i \in \mathcal{N}_{h,l}^C$ and the restrictions was done by the matrix B_l. This section is concerned how to generate the matrix B_l and the gap vector G_l. The matrix B_l and the gap G_l depend on the choice of the master surface l and on the parametrisation.

In Chapter 2 it was noted that there is no rule for choosing the parametrisations $X^{(i)}$. Thus for two possible choices of the parametrisations $X^{(i)}$ the generation of B_l and G_l is explained.

a. From the physical point of view it may be reasonable, that the parametrisation is chosen such that it satisfies the following

$$\|u^R(X)\|_{l_2(\mathbb{R}^d)} = \inf_{X \in \Gamma_C} \left\| \mathcal{P}\left(X^{(1)}(X)\right) - \mathcal{P}(X^{(2)}(X)) \right\|_{l_2(\mathbb{R}^d)} , \tag{7.12}$$

where u should be the unknown solution and $\mathcal{P}(X) := u(X) + X$.
Remember that this choice of parametrisation can result into an ill posed problem. Nevertheless it's a very common choice and gives "good" results.

b. It's also possible to fix the unit vector $n(X)$. This may be done due to geometrical properties of the bodies or the normal vector of the deformed configuration is guessed (or known). In this case the parametrisations $X^{(i)}$ are chosen such that

$$n(X) = \frac{\mathcal{P}\left(X^{(1)}(X)\right) - \mathcal{P}(X^{(2)}(X))}{\left\| \mathcal{P}\left(X^{(1)}(X)\right) - \mathcal{P}(X^{(2)}(X)) \right\|_{l_2(\mathbb{R}^d)}} . \tag{7.13}$$

This kind of choice enables to set n constant and with choosing $X^{(l)}$ as the identity map, the uniform discrete LBB condition becomes trivial.

The domains Ω^i are approximated by triangles and so the matrix B_l and the vector G_l can be calculated by Algorithm 7.1.

For explanation of the used notation $\triangle$, $P_i, \ldots$ see Figure 7.2.

$$\begin{aligned} s\underbrace{(P_2 - P_1)}_{\vec{R}_1} + t\underbrace{(P_3 - P_1)}_{\vec{R}_2} &= P - P_1 - \langle P - P_1, \nu(\triangle)\rangle_{l_2(\mathbb{R}^d)}\, \nu(\triangle) \\ \nu(\triangle) &= \frac{\vec{R}_1 \times \vec{R}_2}{\|\vec{R}_1 \times \vec{R}_2\|_{l_2(\mathbb{R}^d)}} \end{aligned} \tag{7.14}$$

From equation (7.14) it should be obvious how to program the procedure **Barycentric**. The only procedure of Algorithm 7.1 which isn't obvious is the procedure **Parametrisation**. One possibility is to calculate for every surface, generated by the $\triangle \in \mathcal{E}_{h,l-1}^C$, the point Q which fulfils either a.) or b.) and checks whether this point is inside the

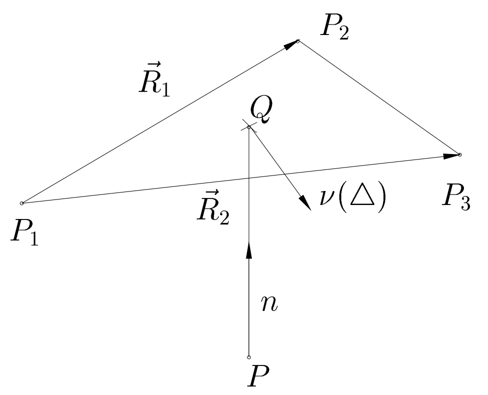


Figure 7.2: barycentric coordinates

Q should be the point in $\triangle$ such that either case a.) or case b.) is fulfilled. $\nu(\triangle)$ is the out warding unit normal vector onto $\triangle$. The barycentric coordinates of Q are determined by equation (7.14)

triangle $\triangle$ or not. For 2D problems this may work fast enough, but for 3D problems this may be rather slow. The time complexity for calculating B_l and G_l is in this case $O(N_l^C N_{l-1}^C) = O(h^{-2(d-1)})$ which isn't optimal in the 3D case. An other possibility to find the triangle $\triangle$ is the use of an ADT (Alternating Digital Tree). This method was presented by Bonet and Peraire [8] and shouldn't be explained in this thesis. In this case the time complexity for generating B_l and G_l reduces (in mean) to $O(N_l^C \ln N_{l-1}^C) = O(h^{-(d-1)} \ln h)$.

Remark 7.11. For numerical computation also *Lagrange* elements of second order were used. In this case the calculation of the **Parametrisation** becomes more difficult. Indeed it's to difficult to calculate the point Q on the curved surface generated by $\triangle$. One simplification would be to divide the triangle $\triangle$ into four parts, such that every node for the ansatz functions becomes a corner point of one of these parts. This simplification is calculate able with a reasonable effort. Nevertheless it's enough to consider only the triangle $\triangle$ itself to achieve a similar accuracy as for the partition of the triangle and thus this was implemented by the author.

PROCEDURE Generate(B, G, l, case)
BEGIN

  WHILE $P \in \mathcal{N}_{h,l}^C$ **DO**
  BEGIN
    $(*$ find $\triangle \in \mathcal{E}_{h,l-1}^C$ such that $*)$
    IF *case* $=$ *a.)* **THEN**
    BEGIN
      $(*$ a.) $\exists\, Q \in \triangle$: $\quad Q \;=\; \underset{\substack{\triangle \in \mathcal{E}_{h,l-1}^C \\ X \in \triangle}}{\operatorname{argmin}} \|\mathcal{P}(X^{(l)}(P)) - \mathcal{P}(X^{(l-1)}(X))\|_{l_2(\mathbb{R}^d)}$ $*)$
      Parametrisation$(P, Q, \triangle, a.))$
    END
    IF *case* $=$ *b.)* **THEN**
    BEGIN
      $(*$ b.) $\exists\, Q \in \triangle\, \exists \lambda \in \mathbb{R}$: $\quad Q \;=\; P(X^{(l)}(P)) + \lambda n(P)$ $*)$
      Parametrisation$(P, Q, \triangle, b.))$
    END
    $(*$ calculate barycentric coordinates $*)$
    Barycentric$(s\,,\, t\,,\, \triangle\,,\, Q\,,\, P)$
    $(*$ calculate $u(Q)$ via barycentric coordinates $*)$
    $u(Q) = (1 - t - s)u(P_1) + su(P_2) + tu(P_3)$
    IF *case* $=$ *a.)* **THEN**
    BEGIN
      $(*$ a.) calculate n from (2.29) $*)$
      $n(P) = \dfrac{\mathcal{P}(Q) - \mathcal{P}(P)}{\|\mathcal{P}(Q) - \mathcal{P}(P)\|_{l_2(\mathbb{R}^d)}}$
    END
    $(*$ set B and G $*)$
    $B_{P,P} = n(P)\,,\; B_{P,P_1} = -(1 - t - s)n(P)\,,\; B_{P,P_2} = -sn(P)$
    $B_{P,P_3} = -tn(P)\,,\; G_P = sign\langle \nu(\triangle), n(P)\rangle_{l_s(\mathbb{R}^d)} \|P - Q\|_{l_2(\mathbb{R}^d)}$
  END

END

Algorithm 7.1: generation of B_l and G_l

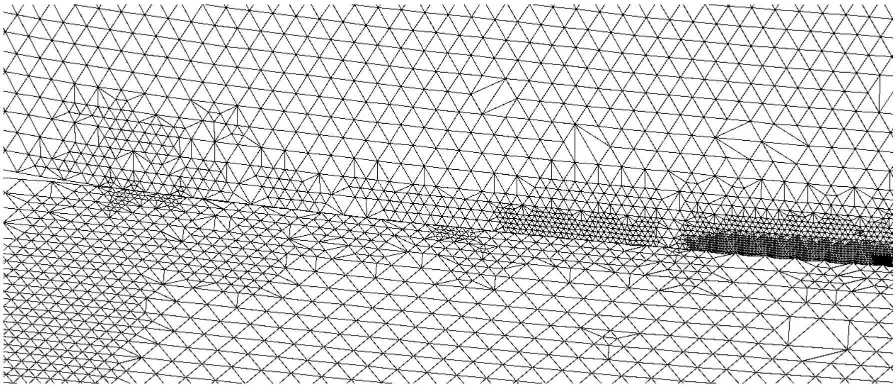


Figure 7.3: mesh generated by $\eta_{T,1}$

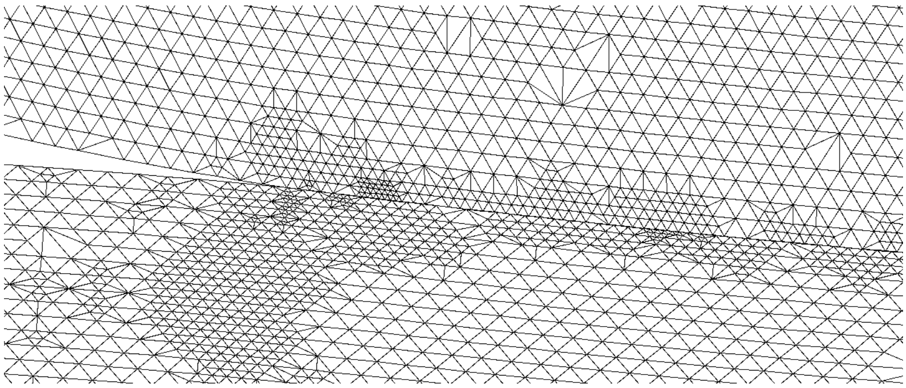


Figure 7.4: mesh generated by $\eta_{T,1} + \eta_{T,2}$

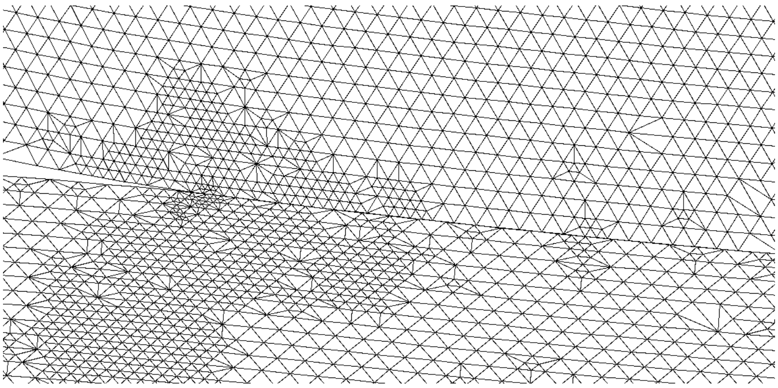


Figure 7.5: mesh generated by stabilised two-sided nodal constraints and $\eta_{T,1}$

Chapter 8

Solving the Body-Body Contact Problem

The contact problem results into a constraint minimisation problem (CMP). A lot of numerical algorithms are developed to handle such CMP. Some of the classical algorithms are explained in GLOWINSKI [26]. In WRIGGERS [51] some algorithms especially for the contact problem are presented. In this work also some approximations for the contact condition are discussed. CARSTENSEN, SCHERF AND WRIGGERS [15] gave a complete numerical analysis for the contact problem of elastic bodies. In the last decade a lot of algorithms are published, based on iterative methods, like HACKBUSCH AND MITTELMANN [27], HOPPE AND KORNHUBER [32], HOPPE [33], BRANDT AND CRYER [11], MANDEL [40], KORNHUBER [36] [37], TARVAINEN [47] [48], DOSTAǍL, GOMES NETO AND SANTOS [20], SCHÖBERL [43].

The algorithm which is presented in this thesis is as simple and old, as it's new. In Section 8.1 an simple algorithm for solving CMP is presented in a very abstract way. For this algorithm convergence results in the A-energy norm are presented. An concrete example of this algorithm is presented in Section 8.2. This example will be the basis for solving the body-body contact problem. To achieve a time optimal algorithm for solving the body-body contact problem some transformations has be done. This transformations is presented in Section 8.3. For an efficient solving an equivalent preconditioner for the transformed body-body contact problem are needed. This preconditioner is a result of the *Bramble Pasciak* transformation, which is presented in Section 8.4. As a consequence of all tools presented up to now, the final time optimal solving algorithm for the body-body contact problem follows. The final algorithm and the proof of its optimal time complexity is presented in Section 8.5. Finally some practical improvements of the solving algorithm are presented in Section 8.6.

8.1 The Approximate Projection Method

A lot of applications result into a CMP, especially the body-body contact problem is a CMP. To solve such a problem numerically a CMP solver is needed. In this section such a solver is developed in an abstract setting. In Section 8.5 this algorithm is applied to the body-body contact problem such that the resulting algorithm has optimal time complexity. Consider the CMP

$$\begin{aligned} J(v) &= \frac{1}{2}\langle Av, v\rangle_{V'\times V} - \langle F, v\rangle_{V'\times V} \\ u &= \operatorname*{argmin}_{v\in K} J(v)\,, \end{aligned} \tag{8.1}$$

with $A : V \to V'$ a symmetric, positive definite and bounded linear operator, $F \in V'$ and K some closed convex set. Note that an operator is called positive definite (spd) iff

$$\langle Av, v\rangle_{V'\times V} > 0 \qquad \forall\, v \in V/\{0\}\,. \tag{8.2}$$

The quadratic functional $J(v)$ is called Ritz functional. Usually A is a matrix resulting from a FEM discretisation. Note that (8.1) has a unique solution because A is spd (Theorem 4.11). Assume that $\hat{A} : V \to V'$ is some spectral equivalent preconditioner of A, i.e. there are positive constants $\underline{\alpha}$, $\overline{\alpha}$, such that

$$\underline{\alpha}\,\langle \hat{A}v, v\rangle_{V'\times V} \le \langle Av, v\rangle_{V'\times V} \le \overline{\alpha}\,\langle \hat{A}v, v\rangle_{V'\times V} \qquad \forall\, v \in V\,.$$

In the following spectral equivalence is denoted by

$$\underline{\alpha}\,\hat{A} \le A \le \overline{\alpha}\,\hat{A}\,, \tag{8.3}$$

or using $\preceq$, $\succeq$, $\simeq$ to reduce writing.
Note that in a lot of applications $\underline{\alpha}$, $\overline{\alpha}$ are independent of the mesh parameter h, but in Section 8.2 there is one application with $\underline{\alpha}$, $\overline{\alpha}$ mesh dependent.

To solve the CMP numerically, a *projected, preconditioned richardson* iteration can be applied (see Algorithm 8.1). The parameter τ is some relaxation parameter, which is chosen such that Algorithm 8.1 converges in the $\hat{A}$-energy norm.
Note that $\hat{A}$ was assumed to be spd and thus $\|.\|^2_{\hat{A}} := \langle A., .\rangle_{V'\times V}$ is a norm.

The operator $P^K_{\hat{A}} : V \to K$ is a projection operator onto K with respect to the $\hat{A}$-energy norm ($\|.\|_{\hat{A}}$), i.e.

$$P^K_{\hat{A}}(\tilde{u}) = \operatorname*{argmin}_{v\in K} \|v - \tilde{u}\|_{\hat{A}}\,. \tag{8.4}$$

In Theorem 4.10 it was proved that $P^K_{\hat{A}}$ is unique and *lipschitz* continuous with lipschitz constant 1. Thus it's possible to apply the ordinary theory for the richardson iteration to achieve convergence results and rates. From this it's known that Algorithm 8.1 converges in the $\hat{A}$-energy norm, with convergence rate

$$\rho = \rho(\hat{A} - \tau A) = \|\hat{A} - \tau A\| \le \max\{|1 - \tau\underline{\alpha}|, |1 - \tau\overline{\alpha}|\} < 1 \qquad \forall\, \tau \in]0, \frac{2}{\overline{\alpha}}[\,.$$

PROCEDURE Projection Algorithm$(K, A, \hat{A}, F, \tau, u^1)$
BEGIN

FOR $k = 1, 2, \ldots$ **DO**
$\tilde{u}^k = u^k + \tau \hat{A}^{-1}\left(F - Au^k\right)$
$u^{k+1} = P^K_{\hat{A}}\left(\tilde{u}^k\right)$

END

Algorithm 8.1: projected preconditioned richardson

Usually the $\hat{A}$-energy norm isn't from further interest, because this norm isn't fast computable and has no physical interpretation. More interesting would be a convergence result in the A-energy norm, at least for the application considered in this thesis. From the convergence in the $\hat{A}$-energy norm the converges in the A-energy norm is deduced, but this convergence may not be monotone. SCHÖBERL [43] proved a monotone decay of the quadratic Ritz functional $J(v)$ and gives an estimate of the convergence rate ρ. This decay of the Ritz functional $J(v)$ also forces convergence in the A-energy norm.

Theorem 8.1 (Energy convergence). *Let the relaxation parameter* $\tau \in]0, \frac{1}{\overline{\alpha}}]$, u^k *be a sequence generated by Algorithm 8.1, then the estimate*

$$J(u^{k+1}) \leq \rho J(u^k) + (1-\rho)J(u) \tag{8.5}$$

holds for every $k \in \mathbb{N}$ *with the convergence rate*

$$\rho = 1 - \frac{\tau\underline{\alpha}}{2} < 1\,. \tag{8.6}$$

The error in the A*-energy norm is bounded by*

$$\|u - u^k\|_A^2 \leq 2\rho^{k-1}\left(J(u^1) - J(u)\right). \tag{8.7}$$

Proof. see SCHÖBERL [43, Theorem 1]. □

Define the condition number $\kappa(\hat{A}^{-1}A)$ by

$$\kappa(\hat{A}^{-1}A) := \frac{\overline{\alpha}}{\underline{\alpha}}\,. \tag{8.8}$$

Note that this definition isn't the original one, but it's up to a mesh independent constant equivalent (provided that $\underline{\alpha}$, $\overline{\alpha}$ are close to the best possible spectral constants). Rewriting the convergence rate ρ by setting $\hat{\tau} = \tau\overline{\alpha} \in]0, 1]$ it's deduces that the convergence rate is only dependent on the condition number $\kappa(\hat{A}^{-1}A)$ and $\hat{\tau}$.

$$\rho = 1 - \frac{\hat{\tau}}{2\kappa(\hat{A}^{-1}A)}$$

The convergence rate is small if the spectral bounds $\underline{\alpha}$, $\overline{\alpha}$ are close, i.e. $\kappa(\hat{A}^{-1}A) \simeq 1$. The time to calculate one *projected, preconditioned richardson* step depends on the time to evaluate $\hat{A}^{-1} \times d$, $A \times d$ and the time to calculate the projection $P_{\hat{A}}^{K}(d)$. For FEM systems the application of a multiplication in A is of optimal time complexity because A is usually a spares matrix. Using a multigrid or multilevel preconditioner for $\hat{A}$, then it's possible to get close spectral bounds and one application $\hat{A}^{-1} \times d$ is as fast as an application of $A \times d$. The only thing which is usually very expansive to compute is the projection $P_{\hat{A}}^{K}$. To improve this draw back the exact projection in Algorithm 8.1 is replaced by some approximate projection $\tilde{P}_{\hat{A}}^{K}$ (Algorithm 8.2). If the approximate projection $\widetilde{P}_{\hat{A}}^{K}$ fulfils a certain decay

PROCEDURE Approximate Projection Algorithm$(K, A, \hat{A}, F, \tau, u^1)$
BEGIN

FOR $k = 1, 2, \ldots$ **DO**
$\tilde{u}^k = u^k + \tau \hat{A}^{-1}\left(F - Au^k\right)$
$u^{k+1} = \widetilde{P}_{\hat{A}}^{K}\left(\tilde{u}^k\right)$

END

Algorithm 8.2: approximate projection algorithm

in the $\hat{A}$-energy norm, then it's also possible to proof monotone convergence in the energy functional $J(v)$ and thus convergence in the A-energy norm, for Algorithm 8.2.

Theorem 8.2. *Let the approximate projection $\widetilde{P}_{\hat{A}}^{K}$ fulfil*

$$\|\widetilde{P}_{\hat{A}}^{K}(\tilde{u}^k) - \tilde{u}^k\|_{\hat{A}}^2 \leq \rho_P \, \|u^k - \tilde{u}^k\|_{\hat{A}}^2 + (1 - \rho_P) \, \|P_{\hat{A}}^{K}(\tilde{u}^k) - \tilde{u}^k\|_{\hat{A}}^2 \,, \tag{8.9}$$

with $\rho_P \in [0, 1[$ and let $\tau \in]0, \frac{1}{\overline{\alpha}}]$. Then Algorithm 8.2 has a monotone convergence of the energy functional $J(u^k)$

$$J\left(u^{k+1}\right) \leq \rho J\left(u^k\right) + (1 - \rho) J(u) \,, \tag{8.10}$$

and convergence of u^k in the A-energy norm

$$\|u - u^k\|_A^2 \leq 2\rho^{k-1}\left(J(u^1) - J(u)\right) \,, \tag{8.11}$$

for all $k \in \mathbb{N}$ with the setting

$$\rho \, = \, 1 - \frac{\tau \underline{\alpha}}{2}(1 - \rho_P) \, < \, 1 \,. \tag{8.12}$$

Proof. see Schöberl [43, Theorem 1] ☐

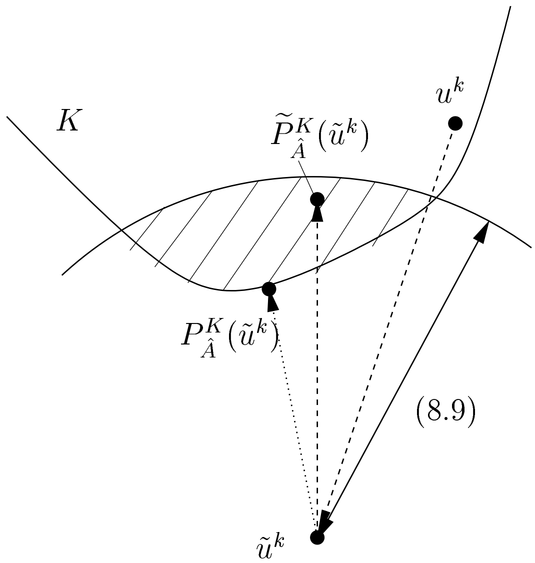


Figure 8.1: sketch of (8.9)

Remark 8.3. Note that $\rho_P = 0$ is nothing else than the exact projection. Condition (8.9) is graphically illustrated in Figure 8.1. The approximate projection operator $\widetilde{P}^K_{\hat{A}}(\tilde{u}^k)$ has to be in the hatched area (Figure 8.1). This area is such that the distance between the point $\tilde{u}^k$, resulting from the preconditioned richardson step, and the new point $u^{k+1} = \widetilde{P}^K_{\hat{A}}(\tilde{u}^k) \in K$ in the hatched area, decreases strict in the $\hat{A}$-energy norm ($\rho_P \in [0, 1[$). If $\rho_P = 0$, i.e. $\widetilde{P}^K_{\hat{A}}$ is the exact projection, this reduction is the best possible.

For practical computation a computable estimate for the iteration error is needed. The next Corollary provides such an estimate, which is nothing else than the error estimator for iterative methods for linear systems.

Corollary 8.4. *Let the sequence u^k be generated by Algorithm 8.2. Then the error of the iteration value u^{k+1} is bounded by*

$$\|u - u^{k+1}\|_A^2 \leq \frac{2\rho}{1-\rho} \langle u^{k+1} - u^k, f - Au^k \rangle_{V' \times V}, \tag{8.13}$$

with ρ from Theorem 8.2.

Proof. see SCHÖBERL [43, Corollary 1]. □

8.2 An Example of an Approximate Projection

Theorem 8.2 guarantees the convergence of the projected preconditioned richardson with an approximate projection $\widetilde{P}^K_{\hat{A}}$, fulfilling (8.9), instead of the exact projection $P^K_{\hat{A}}$. In this section a simple example of an approximate projection, fulfilling (8.9), is presented. To construct such an approximate projection $\widetilde{P}^k_{\hat{A}}$ consider the problem of solving the exact projection $P^K_{\hat{A}}(\tilde{u}^k)$. This problem is nothing else, than solving the quadratic CMP

$$u^{k+1}_{ex} := P^K_{\hat{A}}(\tilde{u}^k) = \operatorname*{argmin}_{v \in K} \|v - \tilde{u}^k\|^2_{\hat{A}}\,, \tag{8.14}$$

which is due to Theorem 4.10 equivalent to the variational inequality

$$\langle \hat{A}u^{k+1}_{ex}, u^{k+1}_{ex} - v\rangle_{V' \times V} \leq \langle \hat{A}\tilde{u}^k, u^{k+1}_{ex} - v\rangle_{V' \times V} \qquad \forall\, v \in K\,.$$

For the preconditioner $\hat{A}$ it was assumed that it's a spd preconditioner and thus there are spectral constant $\underline{\alpha}_2$, $\overline{\alpha}_2$ such that

$$\underline{\alpha}_2 I \;\leq\; \hat{A} \;\leq\; \overline{\alpha}_2 I\,.$$

Note that if A is from an FEM discretisation, the spectral bounds $\underline{\alpha}_2$, $\overline{\alpha}_2$ usually depends on the mesh parameter h and are not close. Nevertheless apply n steps of the exact projection algorithm, i.e. a projected richardson with projection in the V-norm (see Algorithm 8.3).

PROCEDURE Approximate Projection$(K, \hat{A}, n, \tau_2, u^k)$
BEGIN

$w^0 = u^k$
FOR $j = 0, \ldots, n-1$ **DO**
  $\tilde{w}^j = w^j + \tau_2 \hat{A}\left(\tilde{u}^k - w^j\right)$
  $w^{j+1} = P^K_I\left(\tilde{w}^j\right)$
$u^{k+1} = \widetilde{P}^K_{\hat{A}}(\tilde{u}^k) := w^n$

END

Algorithm 8.3: example of an approximate projection

Remark 8.5. The projector P^K_I in Algorithm 8.3 is a L_2-projector. If K is a set of box constraints, then this L_2-projection is trivial.

From Theorem 8.1 it's known that the Ritz functional $J(v) = \|v - \tilde{u}^k\|^2_{\hat{A}}$ for Algorithm 8.3 converges monotone in the $\hat{A}$-energy with a convergence rate $\rho_2 \;=\; 1 - \dfrac{\hat{\tau}_2}{2\kappa(\hat{A})}$, if $\hat{\tau}_2 \;=\; \tau_2\overline{\alpha}_2 \;\in]0,1]$. Thus the Ritz functional decreases after applying Algorithm 8.3 by

$$\|u^{k+1} - \tilde{u}^k\|^2_{\hat{A}} \;\leq\; \rho_2^n \|u^k - \tilde{u}^k\|^2_{\hat{A}} + (1 - \rho_2^n)\,\|u^k_{ex} - \tilde{u}^k\|^2_{\hat{A}}\,.$$

This proves that Algorithm 8.3 is an approximate projection with $\rho_P \;=\; \rho_2^n$.

Remark 8.6. Note that ρ_2 depends on the condition number $\kappa(\hat{A})$ and so ρ_P depends on $\kappa(\hat{A})$ for a fixed n. If A is the matrix generated by a FEM discretisation of an elliptic problem, then $\kappa(\hat{A}) = O(h^{-2})$ (uniform mesh). This isn't very satisfying because Algorithm 8.3 should converge independent from the condition number $\kappa(\hat{A})$ and thus from the mesh parameter h.

To achieve that ρ_P is independent of $\kappa(\hat{A})$ choose

$$n = c\frac{2\kappa(\hat{A})}{\hat{\tau}_2}\,, \tag{8.15}$$

with $c > 0$ is some arbitrary constant. Substituting this n into ρ_P gives

$$\rho_P \;=\; \rho_2^n = \left(1 - \frac{\hat{\tau}_2}{2\kappa(\hat{A})}\right)^{c\frac{2\kappa(\hat{A})}{\hat{\tau}_2}} \;\le\; e^{-c}\,,$$

and thus ρ_P is a constant, invariant of $\kappa(\hat{A})$.

Remark 8.7. That n is depending on the condition number $\kappa(\hat{A})$ is a draw back and it seems so that this approximate projection (Algorithm 8.3) won't be a good idea. Nevertheless in Section 8.5 this approximate projection proves to be the correct one to construct a solver for the body-body contact problem with optimal time complexity.

8.3 Augmented Lagrangian Formulation

In Section 4.2 it was proved that the body-body contact problem is equivalent to the dual formulation, which has the following form.

$$p = \operatorname*{argmin}_{q\in N} \frac{1}{2}\,\langle(\underbrace{BA^{-1}B^* - C}_{=:\,S})q, q\rangle_{Q'\times Q} - \langle q, BA^{-1}F - G\rangle_{Q'\times Q}\,, \tag{8.16}$$

and the displacements $u \in V$ are computed by solving

$$Au + B^*p - F = 0 \quad\Longleftrightarrow\quad u = \operatorname*{argmin}_{v\in V} \|Av + B^*p - F\|_*^2\,. \tag{8.17}$$

The norm $\|.\|_*$ is any norm which is equivalent to $\|.\|_V$. The operator C is either 0 or the operator which is generated by the stabilisation term (7.10). S is called *Schurcomplement*. Problem (8.16) is a quadratic CMP where N has the form of box constraints. Thus it may thought to solve this quadratic CMP with the approximate projection algorithm (Algorithm 8.2). This will work but the algorithm which results has no optimal time complexity, because the *Schurcomplement* isn't sparse. The main problem is the exact inversion of the operator A. This problem is omitted by reformulating the CMP.

Let $\hat{A}$ be some (scaled) spd preconditioner such that

$$\hat{A} < A \leq \overline{\gamma}\hat{A}\,, \tag{8.18}$$

define the *inexact Schurcomplement*

$$S_{in} := B\hat{A}^{-1}B^* + C\,,$$

and let $\hat{S}$ be some *Schurcomplement* preconditioner

$$\underline{\beta}\hat{S} \leq S \leq \overline{\beta}\hat{S}\,.$$

Note that due to the strict inequality $\hat{A} < A$ the norm $\|.\|_{\hat{A}^{-1}-A^{-1}}$ is equivalent to the norm $\|.\|_V$. The reformulation is done by adding zero to the quadratic functional (8.16), indeed add

$$\frac{1}{2}\inf_{v\in V} \|Av + B^*q - F\|^2_{\hat{A}^{-1}-A^{-1}}\,.$$

Note that this infinum attains its limit value zero for all $q \in Q$, because A is V-elliptic and thus invertible (Theorem 4.11). Thus the problem (8.16) and (8.17) are equivalent to

$$\begin{aligned}(u,p) = \operatorname*{argmin}_{(v,q)\in V\times N}\ & \tfrac{1}{2}\langle (BA^{-1}B^* + C)q, q\rangle_{Q'\times Q} - \langle q, BA^{-1}F - G\rangle_{Q'\times Q} \\ & \tfrac{1}{2}\|Av + B^*q - F\|^2_{\hat{A}^{-1}-A^{-1}}\,.\end{aligned} \tag{8.19}$$

Some elementary calculations prove that (8.19) can be written as the quadratic CMP

$$\mathcal{U} = \operatorname*{argmin}_{\mathcal{V}\in V\times N} \mathcal{J}(\mathcal{V}),$$

with

$$\mathcal{J}(\mathcal{U}) = \frac{1}{2}\langle \mathcal{A}\mathcal{U}\,, \mathcal{U}\rangle - \langle \mathcal{F}, \mathcal{U}\rangle\,.$$

The spd operator $\mathcal{A}$ and the functional $\mathcal{F}$ are known from the *Bramble Pasciak* transformation as

$$\begin{aligned}\mathcal{A} &= \begin{pmatrix} (A-\hat{A})\hat{A}^{-1}A & (A-\hat{A})\hat{A}^{-1}B^* \\ B\hat{A}^{-1}(\hat{A}-A) & B\hat{A}^{-1}B^* + C\end{pmatrix} \\ \mathcal{F} &= \begin{pmatrix} (A-\hat{A})\hat{A}^{-1}F \\ B\hat{A}^{-1}F - G\end{pmatrix} \qquad \mathcal{U} = \begin{pmatrix} u \\ p\end{pmatrix}.\end{aligned} \tag{8.20}$$

This formulation seems to be much more complicated than the original one, nevertheless it's the correct one for constructing a time optimal algorithm.

8.4 A Review on Bramble Pasciak Transformation

It was mentioned before that (8.20) is well known from the *Bramble Pasciak* transformation. This transformation was originally applied to an matrix equation, where the matrix was an special indefinite matrix. This indefinite matrix is achieved by setting $N = Q$ in (8.16) and reformulating this minimisation problem into a matrix equation. This equation reads as follows.

$$\begin{pmatrix} A & B^* \\ B & -C \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix} \tag{8.21}$$

For this indefinite system *Bramble and Pasciak* introduced an transformation, which transforms it into a spd system. The transformation is nothing else than the multiplication of the indefinite system (8.21) by the operator

$$\begin{pmatrix} A - \hat{A} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ B & -I \end{pmatrix} \begin{pmatrix} \hat{A}^{-1} & 0 \\ 0 & I \end{pmatrix} .$$

The result of this multiplication is also the spd system

$$\mathcal{A}\mathcal{U} = \mathcal{F} ,$$

with $\mathcal{A}$ and $\mathcal{F}$ presented in (8.20).

The main result of Bramble and Pasciak was the spectrally equivalence of $\mathcal{A}$ to some block diagonal matrix $\hat{\mathcal{A}}$.

Theorem 8.8. *Let $\hat{A}$ be some scaled preconditioner for A (8.18), then the block diagonal operator*

$$\hat{\mathcal{A}} = \begin{pmatrix} \overline{\gamma}(A - \hat{A}) & 0 \\ 0 & B\hat{A}^{-1}B^* + C \end{pmatrix} \tag{8.22}$$

is spectrally equivalent to the block operator $\mathcal{A}$, defined in (8.20), with spectral equivalence constants

$$\underline{\alpha} = 1 - \sqrt{1 - \frac{1}{\overline{\gamma}}} \qquad \overline{\alpha} = 1 + \sqrt{1 - \frac{1}{\overline{\gamma}}} .$$

Proof. see Bramble and Pasciak [9] or Schöberl [43, Theorem 2]. □

Remark 8.9. $\overline{\alpha} \leq 2$ and $\underline{\alpha} > \frac{1}{\overline{\gamma}} = \frac{1}{\kappa(\hat{A}^{-1}A)}$. Thus the condition number $\kappa(\hat{\mathcal{A}}^{-1}\mathcal{A}) \leq 4\kappa(\hat{A}^{-1}A)$. The upper bound 2 for $\overline{\alpha}$ provides a simple choice for the relaxation parameter τ in Algorithm 8.2 and because of the upper estimate of the condition number $\kappa(\hat{\mathcal{A}}^{-1}\mathcal{A})$ by $4\kappa(\hat{A}^{-1}A)$ the convergence rate of Algorithm 8.2 only depends on ρ_P and $\kappa(\hat{A}^{-1}A)$.

Remark 8.10. The reason that the *Bramble Pasciak* transformation wasn't applied directly to the mixed system of the body-body contact problem is that it was not obvious that this transformation preserves the inequality sign.

Remark 8.11. It's obvious that the operator C can also be replaced by an spectral equivalent one. Indeed it's possible to replace the *inexact Schurcomplement* $S_{in} = B\hat{A}^{-1}B^* + \hat{C}$ by some spectral equivalent Schurcomplement preconditioner $\hat{S}$, keeping in mind that then $\underline{\alpha}$, $\overline{\alpha}$ modifies!

8.5 Contact Algorithm Based on Neumann DD

In Section 8.3 the body-body contact problem was reformulated into the quadratic CMP

$$\mathcal{U} = \underset{\mathcal{V}\in V\times N}{\operatorname{argmin}} \frac{1}{2} \langle \mathcal{A}\mathcal{U} \, , \mathcal{U} \rangle - \langle \mathcal{F} , \mathcal{U} \rangle \, ,$$

with $\mathcal{A}$, $\mathcal{F}$ from (8.20). In Section 8.4 it was proved that (8.22) is a good preconditioner for $\mathcal{A}$. Note that due to Remark 8.11 it's possible to replace the *inexact Schurcomplement* by some spectral equivalent *Schurcomplement* preconditioner $\hat{S}$. Now apply Algorithm 8.2, i.e.

i. Apply one preconditioned richardson step

$$\widetilde{\mathcal{U}}^k = \mathcal{U} + \tau \hat{\mathcal{A}}^{-1} \left(\mathcal{F} - \mathcal{A}\mathcal{U}^k \right) .$$

With the notations

$$\begin{aligned} d_u^k &= F - Au^k - B^*p^k \\ w_u^k &= \hat{A}^{-1} d_u^k \\ w_p^k &= B^* w_u^k - \left(G - Bu^k + Cp^k \right) , \end{aligned} \tag{8.23}$$

the richardson step simplifies to

$$\begin{pmatrix} \tilde{u}^k \\ \tilde{p}^k \end{pmatrix} = \begin{pmatrix} u^k \\ p^k \end{pmatrix} + \tau \begin{pmatrix} w_u^k \\ \hat{S}^{-1} w_p^k \end{pmatrix} .$$

It seems so that the inverse of the *Schurcomplement* preconditoner is needed. This is usually very expansive, because only the application of $\hat{S}$ may be cheap. But there is still one step missing

ii. Apply an approximate projection

$$\mathcal{U}^{k+1} = \widetilde{P}_{\hat{\mathcal{A}}}^{V\times N} \left(\widetilde{\mathcal{U}}^k \right) .$$

Because the preconditoner $\hat{\mathcal{A}}$ is block diagonal, the primal and the dual variable decouples. The primal component is unrestricted and thus the projection for it is trivial. Therefor the approximate projection acts only on the dual component. For understanding the approximate projection of the dual component and its advantage, consider first an exact projection, i.e. solve

$$p_{ex}^{k+1} = \underset{q\in N}{\operatorname{argmin}} \|q - \tilde{p}^k\|_{\hat{S}}^2 \,. \tag{8.24}$$

Note that $\|q - \tilde{p}^k\|_{\hat{S}}^2 = \|q\|_{\hat{S}}^2 - 2\langle \hat{S}\tilde{p}^k, q\rangle_{Q'\times Q} + \|\tilde{p}^k\|_{\hat{S}}^2$, and with (8.23) the quadratic functional becomes

$$\|q - \tilde{p}^k\|_{\hat{S}}^2 = \|q\|_{\hat{S}}^2 - 2\langle \hat{S}p^k + \tau w_p^k, q\rangle_{Q'\times Q} + O(1)\,.$$

The last calculation proves that the term including $\hat{S}^{-1}$ only occurs in the constant, which doesn't influence the calculation of p_{ex}^{k+1}. Thus every algorithm, which is based on solving (8.24) approximatively and fulfils (8.9) is an approximate projection such that $\hat{S}^{-1}$ never occurs. Especially Algorithm 8.3 is based on solving (8.24) and is thus a candidate for the approximate projection.

PROCEDURE Augmented Projection Algorithm$(N, \hat{A}, \hat{S}, A, B, C, F, G, u, p)$
BEGIN

  WHILE *not termination criteria* (8.25) **DO**
  BEGIN
    $d_u^k = F - Au^k - B^* p^k$
    $w_u^k = \hat{A}^{-1} d_u^k$
    $w_p^k = B w_u^k - \left(G - Bu^k + Cp^k\right)$
    (* calculate approximated projection $p^{k+1} = \widetilde{P}_{\hat{S}}^N(\hat{S}p^k + \tau w_p^k)$ *)
    ApproximateProjection$(N, \hat{S}, \hat{S}p^k + \tau w_p^k, p^{k+1})$
    (* update u^{k+1} additive or multiplicative *)
    $u_{add}^{k+1} = u^k + \frac{\tau}{\gamma} w_u^k$
    $u_{mul}^{k+1} = u_{add}^{k+1} + \frac{\tau}{\gamma}\hat{A}^{-1} B^* \left(p^k - p^{k+1}\right)$
  END

END

Algorithm 8.4: approximate augmented projection

If the *Schurcomplement* preconditioner is the *inexact Schurcomplement* $\hat{S} = S_{in}$ and thus, $\overline{\alpha} \le 2$, then it's possible to choose $\tau = \frac{1}{2}$ explicitely. A better choice will be

$$\tau = \frac{1}{\overline{\alpha}} = \frac{1}{1 + \sqrt{1 - \dfrac{1}{\gamma}}}\,,$$

if $\overline{\gamma}$ is available.

The iteration error can be estimated by Corollary 8.4, which provides a termination criteria

$$\left\| \begin{pmatrix} u - u^{k+1} \\ p - p^{k+1} \end{pmatrix} \right\|_{\mathcal{A}}^2 \leq c \left[\langle w_u^k, d_u^k \rangle_{V' \times V} + \langle p^{k+1} - p^k, w_p^k \rangle_{Q' \times Q} \right] , \tag{8.25}$$

where c only depends on ρ, $\underline{\alpha}$, $\overline{\alpha}$.

Remark 8.12. If $\mathcal{A}$ is from a standard multigrid or multilevel algorithm, then the *inexact Schurcomplement* isn't sparse. Thus Algorithm 8.4 hasn't optimal time complexity. But due to Remark 8.11 it's possible to replace the *inexact Schurcomplement* with some other spectral equivalent *Schurcomplement* preconditioner. Note that in this case the spectral equivalence constants $\underline{\alpha}$, $\overline{\alpha}$ are not determined by Theorem 8.8 and thus the choice of τ isn't known a-priori. The *Schurcomplement* preconditoner can be constructed such that it's sparse i.e. one multiplication with $\hat{S}$ is of time complexity N_l^C.

Example 8.1. At the beginning of this chapter it was claimed that the solving algorithm for the body-body contact problem is of optimal time complexity. This is proved for the special case that the approximate projection is calculated by Algorithm 8.3, that $\hat{S}$ is some sparse *Schurcomplement* preconditioner, that the mesh which generates A, B is uniform and that the calculation is without the stabilising term C. Essential for the time complexity of Algorithm 8.3 was the condition number, i.e. $\kappa(\hat{S})$. The next lemma provides an estimate of the condition number $\kappa(BA^{-1}B^*)$ and thus for $\kappa(\hat{S})$.

Lemma 8.13. *let A be V-elliptic and bounded, B be given by $.\frac{R}{N}$ with n constant and $X^{(l)}$ the identity map, then condition number of the operator $BA^{-1}B^*$ for an uniform mesh is $\kappa(BA^{-1}B^*) = O(h_l^{-1})$.*

Proof. Let $q_h \in Q_{h,l}$ and consider $\langle BA^{-1}B^* q_h, q_h \rangle_{Q' \times Q} = \langle A^{-1}B^* q_h, B^* q_h \rangle_{Q' \times Q}$. Define u_h as the solution of the variational equation

$$\langle A u_h, v_h \rangle_{V' \times V} = \langle B^* q_h, v_h \rangle_{V' \times V} \qquad \forall\, v_h \in V_h \,.$$

Thus the term $\langle BA^{-1}B^* q_h, q_h \rangle_{Q' \times Q}$ becomes

$$\langle BA^{-1}B^* q_h, q_h \rangle_{Q' \times Q} = \langle A u_h, u_h \rangle_{V' \times V} = \|u_h\|_A^2 \,.$$

Because A is V-elliptic and bounded, the A-energy norm of u_h can be represented as

$$\|u_h\|_A = \sup_{w_h \in V_h} \frac{\langle A u_h, w_h \rangle_{V' \times V}}{\|w_h\|_V} \simeq \sup_{w_h \in V_h} \frac{\langle B w_h, q_h \rangle_{Q' \times Q}}{\|w_h\|_V} \,. \tag{8.26}$$

This term can be estimated from above due to the continuity of B via

$$\|u_h\|_A \overset{(8.26)}{\simeq} \sup_{w_h \in V_h} \frac{\langle B w_h, q_h \rangle_{Q' \times Q}}{\|w_h\|_V} \preceq \|q_h\|_Q \leq \|q_h\|_{L_2(\Gamma_C)} \,.$$

The estimate from below is more tricky. First note that

$$\|g_h\|_Q \simeq \inf_{\substack{w\in V_l\times\{0\}\\ Bw=g_h}} \|w\|_V \simeq \inf_{\substack{w_h\in V_{h,l}\times\{0\}\\ Bw_h=g_h}} \|w_h\|_V . \tag{8.27}$$

$$\begin{aligned}
\|u_h\|_A &\overset{(8.26)}{\simeq} \sup_{w_h\in V_h} \frac{\langle Bw_h, q_h\rangle_{Q'\times Q}}{\|w_h\|_V} \geq \sup_{w_h\in V_{h,l}\times\{0\}} \frac{\langle Bw_h, q_h\rangle_{Q'\times Q}}{\|w_h\|_V} \\
&= \sup_{g_h\in \mathrm{tr}\, V_{h,l}} \sup_{\substack{w_h\in V_{h,l}\\ Bw_h=g_h}} \frac{\langle Bw_h, q_h\rangle_{Q'\times Q}}{\|w_h\|_V} = \sup_{g_h\in \mathrm{tr}\, V_{h,l}} \frac{\langle g_h, q_h\rangle_{Q'\times Q}}{\inf_{\substack{w_h\in V_{h,l}\\ Bw_h=g_h}} \|w_h\|_V} \\
&\overset{(8.27)}{\simeq} \sup_{g_h\in \mathrm{tr}\, V_{h,l}} \frac{\langle g_h, q_h\rangle_{Q'\times Q}}{\|g_h\|_Q} \overset{\text{Theorem 5.2}}{\succeq} \sup_{g_h\in \mathrm{tr}\, V_{h,l}} \frac{\langle g_h, q_h\rangle_{Q'\times Q}}{h_l^{-\frac{1}{2}}\|g_h\|_{L_2(\Gamma_C)}} \\
&\overset{(*)}{=} h_l^{\frac{1}{2}} \sup_{g\in Q'} \frac{\langle g, q_h\rangle_{Q'\times Q}}{\|g\|_{L_2(\Gamma_C)}} \frac{\|g\|_{L_2(\Gamma_C)}}{\|I^l g\|_{L_2(\Gamma_C)}} \overset{(*)}{\succeq} h_l^{\frac{1}{2}} \sup_{g\in Q'} \frac{\langle g, q_h\rangle_{Q'\times Q}}{\|g\|_{L_2(\Gamma_C)}} = h_l^{\frac{1}{2}} \|q_h\|_{L_2(\Gamma_C)}
\end{aligned}$$

(*) I^l is a *Fortin* operator. The proof is, due to the assumption n is constant and $X^{(l)}$ is the identity map, similar to the corresponding proof in Proposition 6.15. Summing up both estimates everything is done.

$$h_l\|q_h\|^2_{L_2(\Gamma_C)} \preceq \langle BA^{-1}B^* q_h, q_h\rangle_{Q'\times Q} \preceq \|q_h\|^2_{L_2(\Gamma_C)}$$

□

To prove the optimal time complexity rewrite the algorithm

WHILE *not termination criteria* (8.25) **DO**
BEGIN
$w = \hat{A}^{-1}(F - Au^k - B^*p^k)$
$d = \hat{S}p^k + \tau\left(Bw - (G - Bu^k)\right)$
FOR $j = 0, \dots, n_i - 1$ **DO**

$$p^{k+\frac{j+1}{n_i}} = P_I^N\left(p^{k+\frac{j}{n_i}} + \tau_2(d - \hat{S}p^{k+\frac{j}{n_i}})\right)$$

(∗ update u^{k+1} additive or multiplicative ∗)
$u^{k+1}_{add} = u^k + \frac{\tau}{\gamma} w_u$
$u^{k+1}_{mul} = u^{k+1}_{add} + \frac{\tau}{\gamma}\hat{A}^{-1}B^*\left(p^k - p^{k+1}\right)$
END

- P_I^N is a L_2 projection, which is trivial for box constraints and thus for N.
- n_i large enough ($n_i = O(\kappa(\hat{S}))$) then $\rho_P < 1$ independent of the mesh parameter h (see Section 8.2) and due to termination criteria (8.25) the algorithm is terminated after a finite number (independent of the mesh parameter h) of outer iterations ($n_o = O(\ln\epsilon)$). ϵ denotes the relative iteration error.

- $\kappa(\hat{S}) = O(h_l^{-1})$ (see Lemma 8.13)
- time complexity (uniform refined): $N = O(h^{-d}) \quad N_l^C = O(h_l^{1-d})$

$$n_o(c_o + n_i c_i) = O(\ln \epsilon)\big(O(N) + O(N_l^C)O(h_l^{-1})\big) = O(N \ln \epsilon)\,.$$

  c_o denotes the costs for calculating w, d, u^{k+1} and c_i are the costs for calculating the approximate projection P_I^N and one multiplication of $\hat{S}$, i.e. calculating one time $P_I^N\left(p^{k+\frac{j}{n_i}} + \tau_2(d - \hat{S}p^{k+\frac{j}{n_i}})\right)$.

Remark 8.14. It's not usual that the preconditioner $\hat{A}$ is scaled. If this is not the case, i.e.

$$\underline{\gamma}\hat{A} < A \leq \overline{\gamma}\hat{A}\,,$$

then $\hat{A}$, $\overline{\gamma}$ has to be replaced by $\hat{A} \to \underline{\gamma}\hat{A}$ and $\overline{\gamma} \to \frac{\overline{\gamma}}{\underline{\gamma}}$.

If $\underline{\gamma}$, $\overline{\gamma}$ are not available, then it's possible to calculate them by the Lancos-method, without losing time complexity.

8.6 Practical Improvements

- In Example 8.1 it was proved that Algorithm 8.4 with the approximate projection from Algorithm 8.3 and a sparse *Schurcomplement* preconditioner $\hat{S}$ is optimal in time. One disadvantage of Algorithm 8.3 for the approximate projection is that the relaxation parameter τ_2 has to be chosen and thus $\overline{\beta}$ with $\underline{\beta}\hat{S} \leq S \leq \overline{\beta}\hat{S}$ has to be known. Furthermore it's known that a richardson iteration converges not as fast as other iteration algorithms (e.g. cg-iteration). Thus replace Algorithm 8.3 for the approximate projection by the projection algorithm introduced by Dostal [19]. This algorithm is a cg kind algorithm and thus a faster convergence is expected (not proved). Furthermore a cg algorithm is self scaling, i.e. the relaxation parameter τ_2 isn't necessary any more.
- In Algorithm 8.4 an initial guess (u^0, p^0) is needed. For the first level $(u^0, p^0) = (0, 0)$ is the simplest choice. For higher levels it's possible to choose (u^0, p^0) as in the first level but with this choice information is lost, because the solution of the last level is known. Thus prolongate the solution from the last level to the new one. For the primal variable u this is trivial, because u is a grid function, i.e. it has an interpretation in the continuous space V. The dual variable p is only defined in the nodes at the master surface, or if $C \neq 0$, p is defined in all nodes of both contact surfaces. Assume that the calculation is done without stabilisation, i.e. $C = 0$. Then the *nodal constraints* has an interpretation in the continuous space Q. From Chapter 6 it's known that

$$p_L = \sum_{x_i \in \gamma_{l,L}^C} p_{i,L} I_L^l(.)(x_i)\,.$$

L is the index describing the level and l is the number of the master surface. The prolongation of level $L-1$ to level L has to fulfil

$$\langle p_{L-1}, \varphi_{i,l,L}\rangle_{Q'\times Q} = \langle p_L, \varphi_{i,l,L}\rangle_{Q'\times Q} = p_{i,L} \qquad \forall\, x_i \in \gamma^C_{l,L}\,.$$

Note that $\varphi_{i,l,L}$ is the basis ansatz function in node x_i at level L, restricted to Γ_C. The calculation of $\langle p_{L-1}, \varphi_{i,l,L}\rangle_{Q'\times Q}$ is difficult and it's possible that some values $p_{i,L} < 0$. This kind of prolongation seems to be useless for an initial guess. As an ad hoc guess try the following approach, which gives good results for linear Lagrange elements. Multiply the vector p_{L-1} with the inverse lumped mass matrix (lumped mass matrix see Section 6.5). Now consider p_L as a grid function on Γ^l_C and prolongate it to the next level. The initial value for p^0_L is then nothing else than the prolongated function multiplied with the lumped mass matrix of level L. The advantage of this approach is that every component of $p^0_L \geq 0$. In practical experiments this approach reduces the time in the approximate projection about $\frac{2}{3}$.
For quadratic Lagrange elements the initial guess for $p^0 = 0$ seems to be the safest one.

- From Theorem 8.8 it's known that the spectral constants for the preconditioner $\hat{\mathcal{A}}$, with S_{in} only depend on the condition number $\kappa(\hat{A}^{-1}A)$ and for standard multigrid, multilevel preconditioners this condition number $\kappa(\hat{A}^{-1}A) \simeq 1$, which guarantees the best possible theoretical convergence rates. On the other hand it's known that S_{in} isn't sparse, i.e. one multiplication of S_{in} is of order $O(N)$ and thus it's not possible to get a time optimal algorithm. Using a sparse *Schurcomplement* preconditioner $\hat{S}$ instead of S_{in}, the solver will be of optimal time complexity but the condition number $\kappa(\hat{\mathcal{A}}^{-1}\mathcal{A})$ depends on $\kappa(\hat{A}^{-1}A)$ and $\kappa(\hat{S}^{-1}S)$, which results usually in much worse convergence rates, because it's usually not possible to get $\kappa(\hat{S}^{-1}S)$ as close to one as $\kappa(\hat{A}^{-1}A)$. The aim is to find a compromise at least a practical one.

Remark 8.15. The choice of the preconditioner $\hat{A}$ is standard. For this refer to Bramble [10] or other books about multigrid. The choice of the sparse *Schurcomplement* preconditioner is more tricky. For this refer to Schöberl [43] [43].

PROCEDURE Main()
BEGIN

  IF L *is first level* **THEN**
  BEGIN
    $u^0 = 0\,,\, p^0 = 0$
  END
  ELSE
  BEGIN
    $u^0 =$ **Prolongate**(u_{L-1})
    $p^0 = \overline{M}.$**Prolongate**(p_{L-1})
  END
  (∗ linearise problem (Algorithm 7.1) ∗)
  Generate($B\,,\,G$)
  (∗ solve linearised problem (Algorithm 8.4) ∗)
  AugmentedProjectionAlgorithm($N, \hat{A}, \hat{S}, A, B, C, F, G, u, p$)
  (∗ calculate grid function p ∗)
  $p_L = \overline{M}^{-1} p$

END

Algorithm 8.5: "efficient" body-body contact solver

Chapter 9

Numeric Results

In Chapter 7 a method (*nodal constraints*) was introduced to discretise the body-body contact problem. Also an ad-hoc refinement strategy for the body-body contact problem was presented. For solving the body-body contact problem a time optimal algorithm was presented in Chapter 8. With all these results it's possible to solve the body-body contact problem numerically.

In this section numerical results for both, the solution of academic problems as well as the solution of a real life problem (the sag of a roll stack), are presented. The calculations have been carried out within the C++ finite element code FE++ (Documentation see Schöberl [42]). The calculations of the academic results were done on a SGI Origin 2000, CPU R12000, 300MHz, whereas the calculations of the real life problem were done on a SGI Origin 2000, CPU R10000, 195 MHz.

In the tables presented the following notation is used

Notation:

N..........number of nodes.

N_l^C.......number of nodes at the contact boundary Γ_C^l which is equal to the number of inequalities.

$n_o(\epsilon)$......number of outer iteration to reduce the error by a factor ϵ.

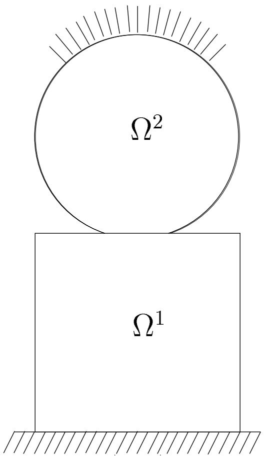
$\overline{n_i}$..........average number of steps for approximate projection

T_{solve}.....total time spent in the solver.

T_{proj}......average time spent in the approximate projector.

$\kappa(\hat{S})$......condition number of the *Schurcomplement preconditioner*

9.1 Contact between a Circle and a Square (2D Problem)



$\Omega^1 = \square(0.5,0.5;0.707)$

$\Omega^2 = \mathrm{C}(0.5,2.5;0.707)$

E=1, ν=0.3

f=(0,0,-0.01)

The calculation was done for linear *Lagrange* elements, for both, a uniform refined mesh (Table 9.1) and an adaptive refined mesh (Table 9.2). The adaptive refined mesh was generated by using the ad-hoc a-posteriori error estimator $\eta_{T,1} + \eta_{T,2}$ presented in Section 7.4.

For the preconditioning of $\hat{A}$ a standard multigrid preconditioner (V-cycle with 1 pre- and 1 post- smoothing step (V 11)) was used. The *Schurcomplement* preconditioner $\hat{S}$ is more tricky, refer to the references given in Remark 8.15.

In Figure 9.1 the time needed for solving the body-body contact problem T_{solve} versus the number of unknowns N is illustrated in a logarithmic scale. From theory (Section 8.5) it's known that the time complexity of the algorithm is linear if $\hat{S}$ is sparse and $\kappa(S) = O(h^{-1})$ (Lemma 8.13). The *Schurcomplement* preconditioner used for this problem should be sparse (Remark 8.15), nevertheless Figure 9.1 shows that the gradient is $\approx \frac{3}{2} > 1$. The reason therfore is that the intern multiplication for one multigrid step isn't correct implemented and thus the *Schurcomplement* preconditioner isn't sparse. In Figure 9.2 the error, measured by the ad-hoc a-posteriori error estimator $\eta_{T,1} + \eta_{T,2}$, versus the number of unknowns N is illustrated. Also the condition number $\kappa(\hat{S})$ versus the number of unknowns is illustrated (see Figure 9.3). For a uniform refined mesh the condition number should be of order h^{-1} (Lemma 8.13). The numeric results show a similar behaviour for the adaptive refined mesh.

For the adaptive refined mesh, the number of contact nodes N_l^C, are larger than for the uniform refined mesh. Furthermore the snap shot of the adaptive refined mesh (Figure 9.4) shows that the mesh becomes finer in the region where the contact takes off. This is what is expected because the solution a the take off area is less smooth.

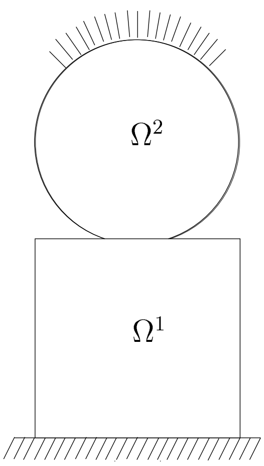
Nodes	N^C	$n_o(10^{-3})$	$\overline{n_i}$	$T_{solve}/[\mathrm{s}]$	$T_{proj}/[\mathrm{s}]$	$\kappa(\hat{S})$
4226	33	19	9	3.3	0.66	21.2
16642	65	19	11	15.96	2.75	40.9
66050	129	20	15	149.08	35.46	79.7
263170	257	21	19	910.67	221.09	158.5

Table 9.1: circle square (2D) uniform refined

Nodes	N^C	$n_o(10^{-3})$	$\overline{n_i}$	$T_{solve}/[\mathrm{s}]$	$T_{proj}/[\mathrm{s}]$	$\kappa(\hat{S})$
6800	69	18	22	21.40	7.13	71.8
9522	85	18	29	37.23	13.27	94.9
14443	119	19	31	66.02	25.03	136.0
32418	174	19	33	124.26	47.53	190.6
55196	257	20	37	269.15	105.46	308.8
110652	418	21	44	753.17	338.73	503.2

Table 9.2: circlesquare (2D) adaptive refined ($\eta_1 + \eta_2$)

9.2 Contact between a Cube and a Sphere (3D Problem)



$\Omega^1 = \square(0, 0, 0; 1)$

$\Omega^2 = S(0.5, 0.5, 2.5; 0.866)$

E=1, ν=0.3

f=(0,0,-0.01)

The calculation was done for linear *Lagrange* elements, for both, a uniform refined mesh (Table 9.3) and an adaptive refined mesh (Table 9.4). The adaptive refined mesh was generated by using the ad-hoc a-posteriori error estimator $\eta_{T,1} + \eta_{T,2}$ presented in Section 7.4.

For the preconditioning of $\hat{A}$ a standard multigrid preconditioner (V-cycle with 1 pre-

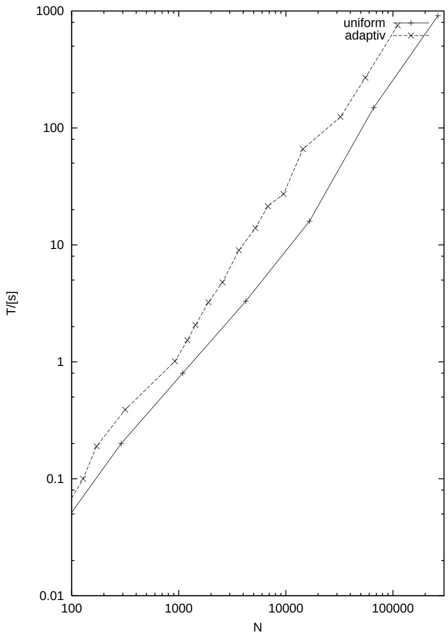


Figure 9.1: $N - T_{solve}$ circle square (2D)

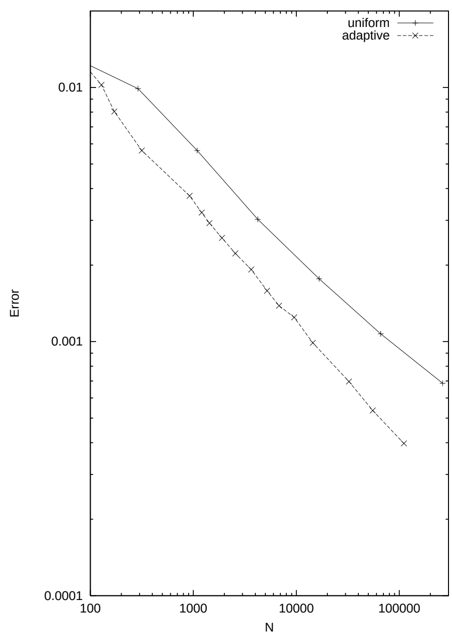


Figure 9.2: $N - \max_{T \in \tau_h}(\eta_{T,1} + \eta_{T,2})$ circle square (2D)

and 1 post- smoothing step (V 11)) was used. The *Schurcomplement* preconditioner $\hat{S}$ is more tricky, refer to the references given in Remark 8.15.

In Figure 9.6 the time needed for solving the body-body contact problem T_{solve} versus the number of unknowns N is illustrated in a logarithmic scale. From theory (Remark 8.15) it's known that the time complexity of the algorithm is linear if $\hat{S}$ is sparse and $\kappa(S) = O(h^{-1})$ (Lemma 8.13). The *Schurcomplement* preconditioner used for this problem should be sparse (Remark 8.15), nevertheless Figure 9.6 shows that the gradient is $\approx \frac{3}{2} > 1$. The reason therefore is that the intern multiplication for one multigrid step isn't correct implemented and thus the *Schurcomplement* preconditioner isn't sparse. In Figure 9.7 the error, measured by the ad-hoc a-posteriori error estimator $\eta_{T,1} + \eta_{T,2}$, versus the number of unknowns N and in Figure 9.8 the condition number $\kappa(\hat{S})$ versus the number of unknowns is illustrated. For the uniform mesh the condition number $\kappa(\hat{S})$ increases like it was proved in Lemma 8.13.

9.3 Sag of a Roll Stack (3D)

The VAi Linz produces sheet metal. To produce sheet metal of high quality for automobile industry it's necessary to get uniform thin metal. The production of sheet metal is done by putting metal into a rolling mill. One roll by itself has a weight of about 200 tons. The rolling sheet causes forces on the rolling mill of about 30 000kN. Because of the large scale

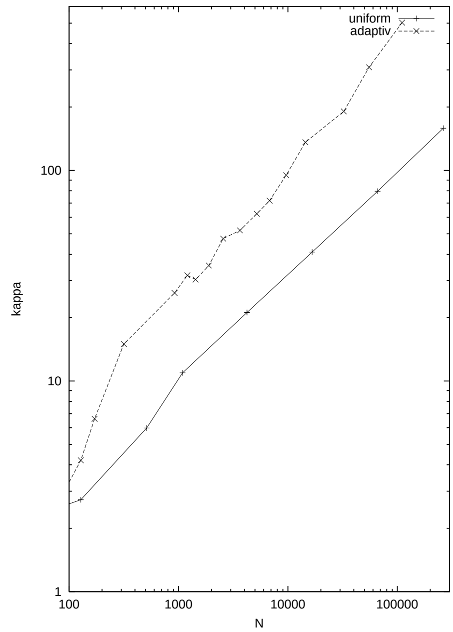


Figure 9.3: $N - \kappa(\hat{S})$ circle square (2D)

Nodes	N_C	$n_o(10^{-2})$	$\overline{n_i}$	$T_{solve}/[\mathrm{s}]$	$T_{proj}/[\mathrm{s}]$
2482	81	107	8	15.29	5.09
18018	289	112	26	254.38	94.79
137410	1089	115	28	4782.41	1745.61

Table 9.3: cube sphere (3D) contact uniform refined

(length) of the rolls, it's obvious that the rolls sag and so the thickness of the sheet varies. To prevent or reduce this effect a supporting roll and a working roll is used. In addition the rolls aren't cylindrical, they are bulbous. Also the forces at the bears can be varied to reduce the sag. Because of the high forces, the rolls can't be regarded as rigid, they will deform. To get the sag of the working roll a body-body contact problem (contact between support- and working- roll) has to be solved.

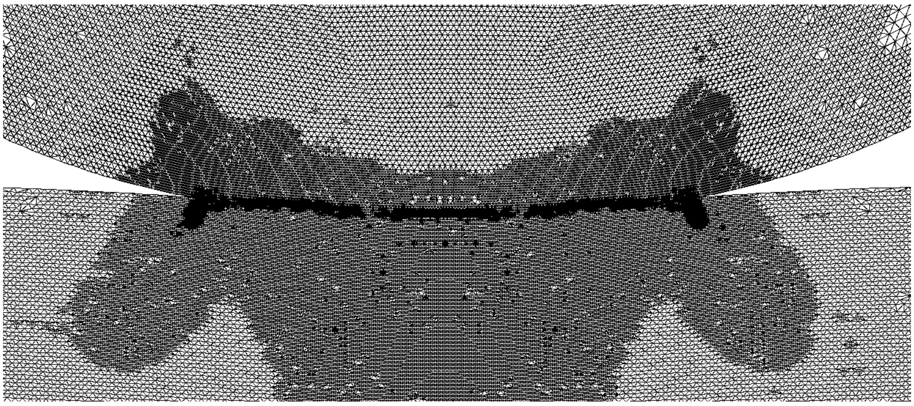
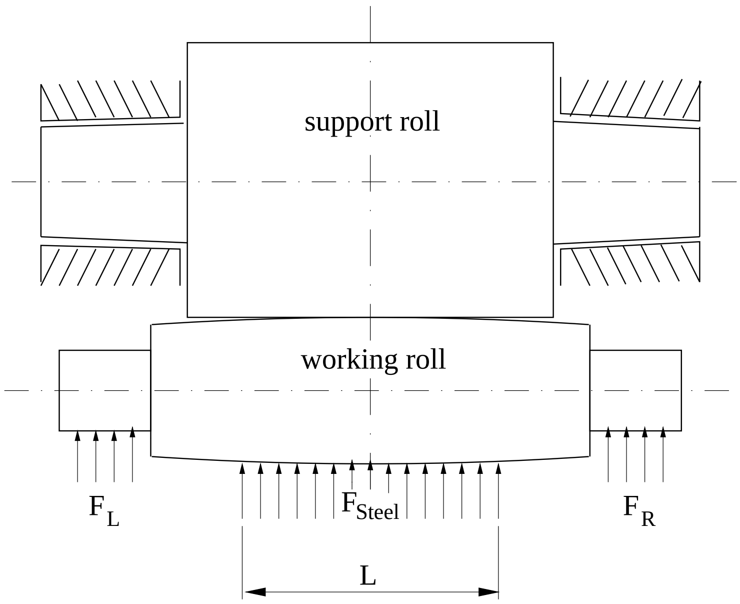


Figure 9.4: snap shot of adaptive refined mesh ($\eta_{T,1} + \eta_{T,2}$)

Nodes	N_C	$n_o(10^{-2})$	$\overline{n_i}$	$T_{solve}/[s]$	$T_{proj}/[s]$
7393	405	55	38	126.74	67.74
11696	485	57	43	334.85	170.62
32773	1112	63	55	1424.06	759.31
85074	1963	70	66	5774.01	4143.81

Table 9.4: cubesphere (3D) contact adaptive refined $\eta_{T,1} + \eta_{T,2}$



but the eigenvalues of the problem may be very small, and so for numerical stability it would be better to regularise. Because of the large scale and the small contact zone a very fine mesh is needed to resolve the contact zone. In the inner of the domain the mesh hasn't to be as fine, because the solution is expected to be more regular there. So an adaptive refinement strategy is chosen. The results which are presented in the following are calculated for two different loads. The Figure 9.15, 9.16 show the sag of the rolls measured along a line. In Figure 9.14 the notation used, is explained. The reason why it was stressed out that the sag is measured along a line, is that due to the bulbous form of the working roll the figures given in 9.15, 9.16 have to be corrected. This is done by the considering the "Schliff". The "Schliff"denotes the difference of the thickness of the working roll. For the this calculations the difference of this thickness is $150\mu m$.

The calculations were done on a R10000/195 MHz processor of an SGI Origin 2000 machine and took about 90 minutes.

The preconditioner $\hat{A}$ was a standard multigrid preconditioner, V-cycle with 3 pre- and 3 post- smoothing steps (V 33). For this calculations the *Schurcomplement* preconditoner was given by the *inexact Schurcomplement* and thus $\hat{S}$ was not sparse. Note that in this case the solving algorithm has no optimal time complexity. Never mind because a sparse *Schurcompement* preconditioner wouldn't work due to the implementation error of the multiplication of the smoothers.

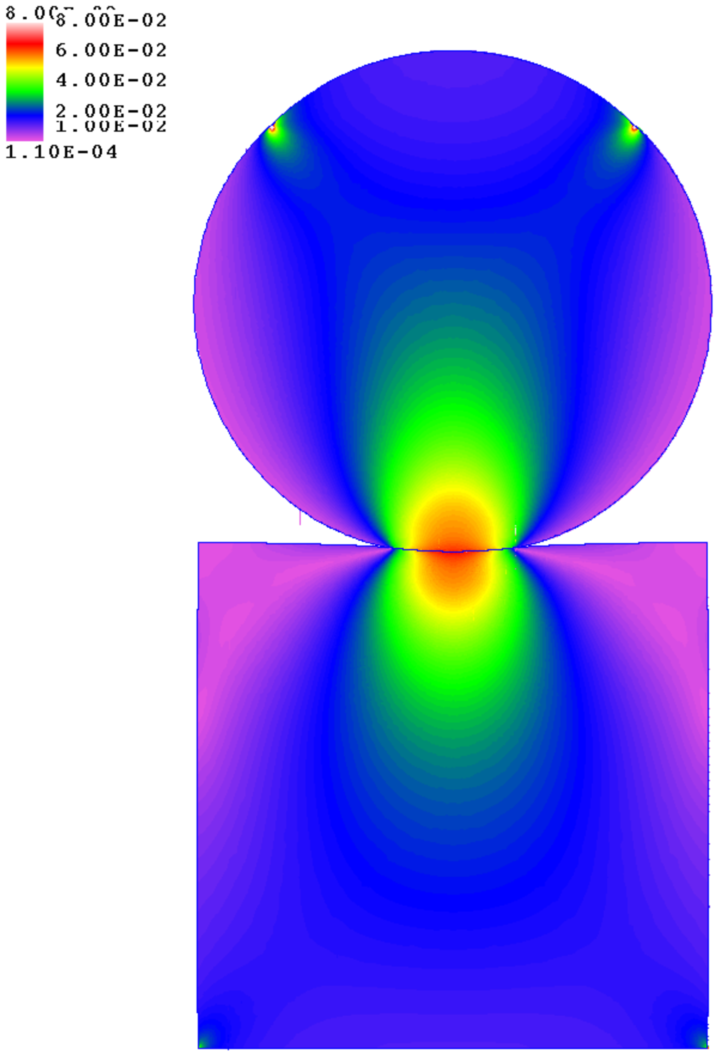


Figure 9.5: van mises stress

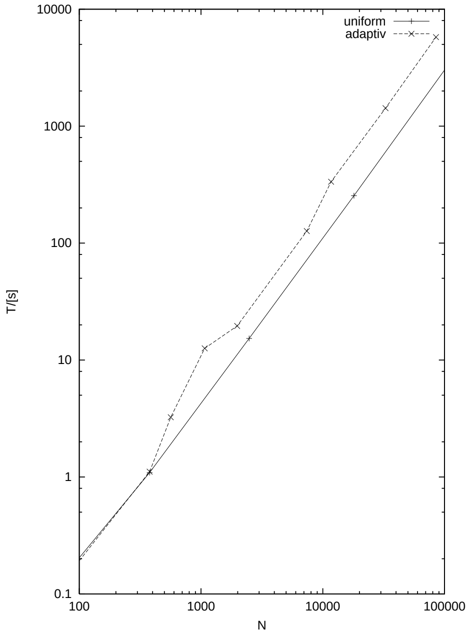


Figure 9.6: $N - T_{solve}$ for cube sphere (3D)

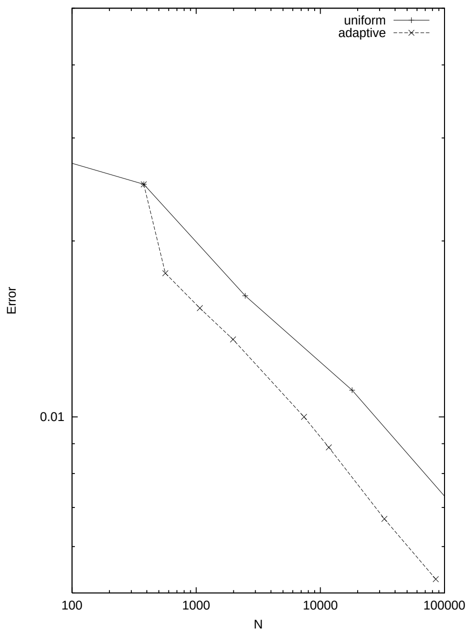


Figure 9.7: $N - \max_{T \in \tau_h}(\eta_{T,1} + \eta_{T,2})$ for cube sphere (3D)

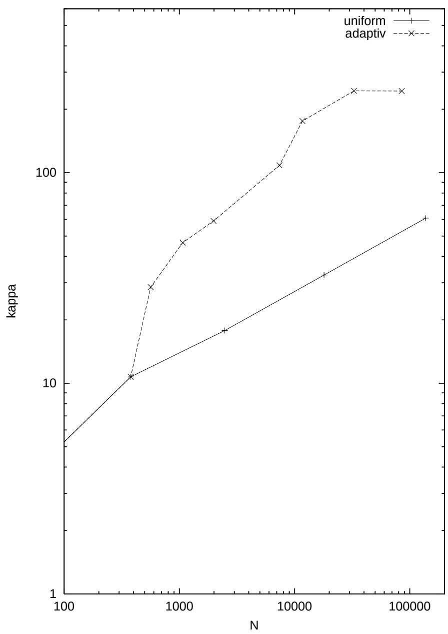


Figure 9.8: $N - \kappa(\hat{S})$ cube sphere (3D)

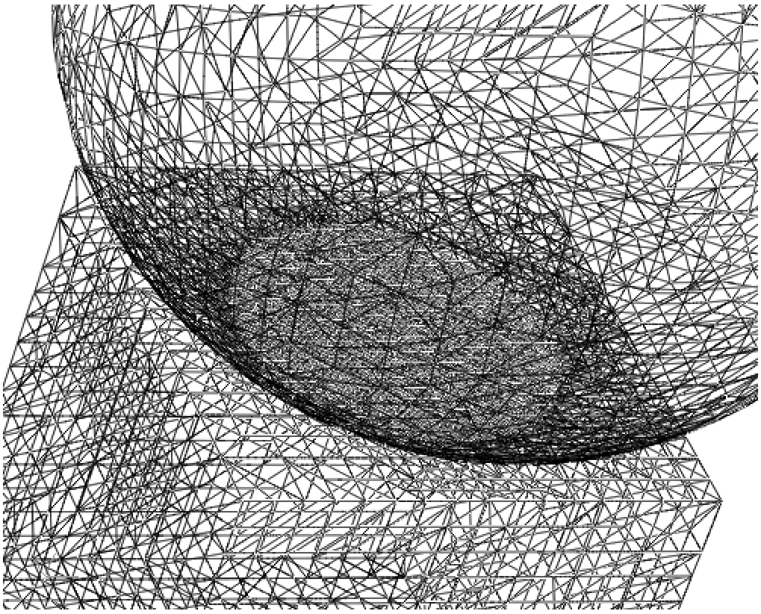


Figure 9.9: surface mesh for adaptive refined mesh $\eta_{T,1} + \eta_{T,2}$

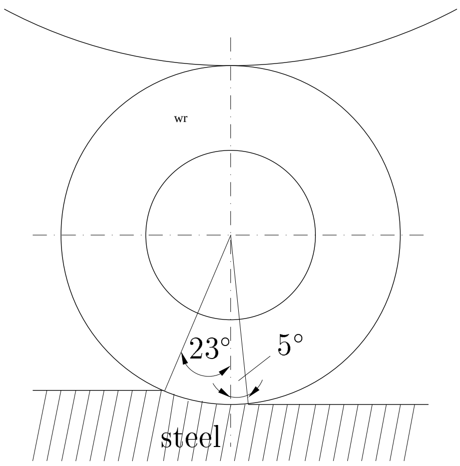


Figure 9.10: distribution of the steel pressure along the working roll

Schliff = 150μm

E=220, ν=0.3

f=(0,0,-76.6$10^{-6}$)

p=(0,0,F_{steel})

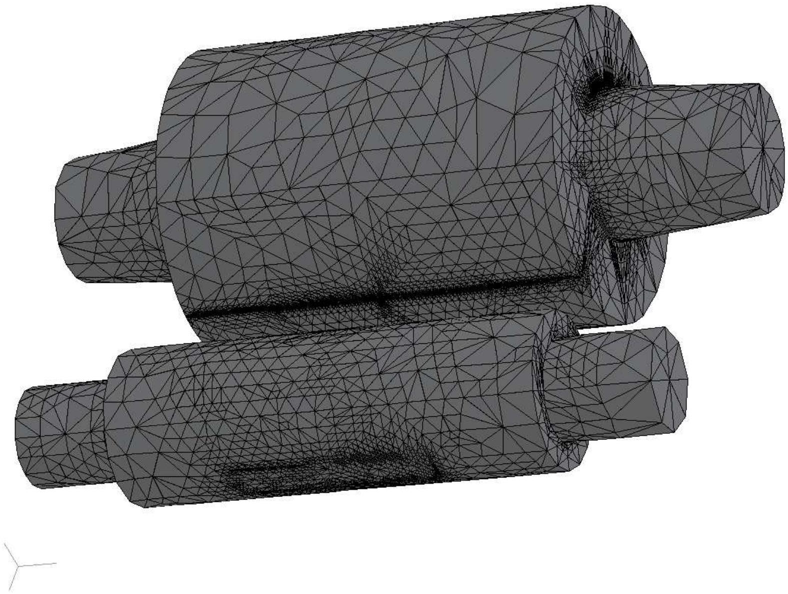


Figure 9.11: mesh for $F_{\text{steel}} = 15\text{KN}$, $L = 1m$ Nodes: 46611, Tet.:246338, Inequalities: 7089

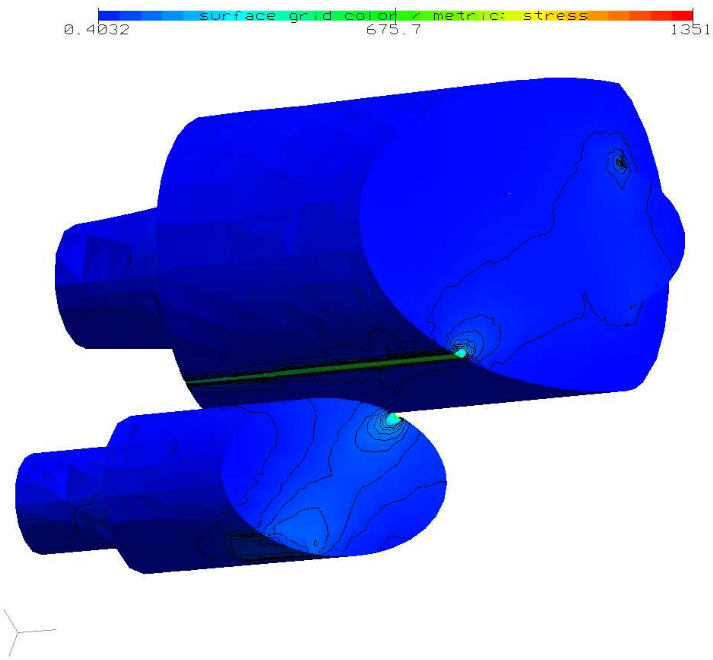


Figure 9.12: Van Mises stress along a cut surface. $F_{\text{steel}} = 15\text{KN}$, $L = 1m$

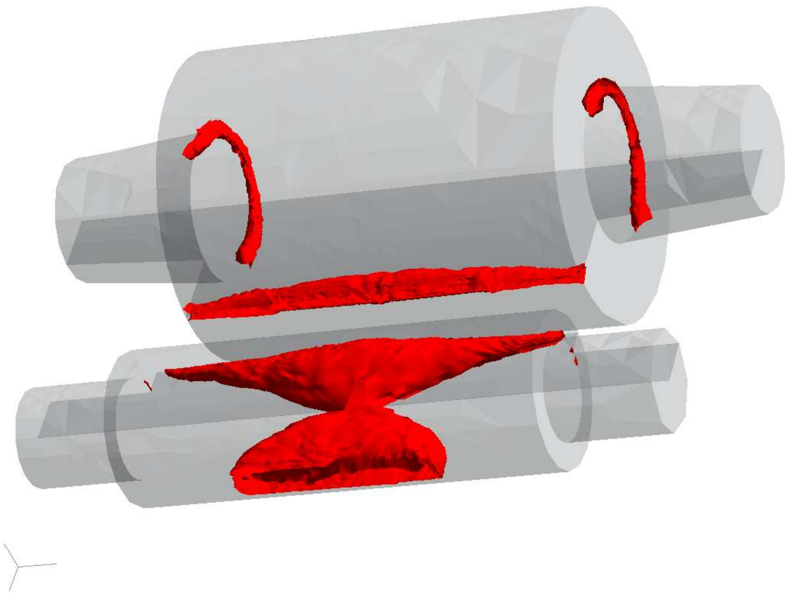


Figure 9.13: Isocline for $F_{steel} = 15$KN, $L = 1m$

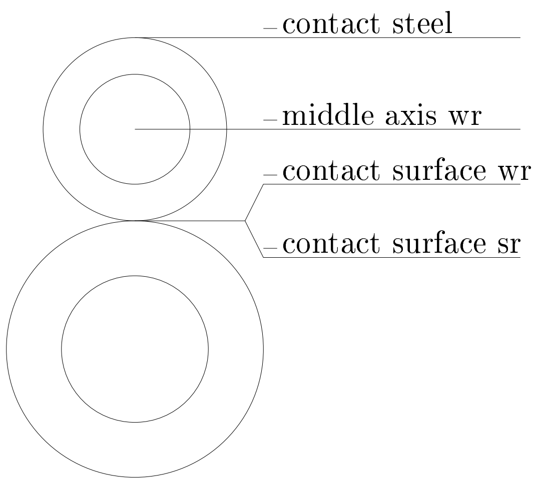


Figure 9.14: notation for Figure 9.15+ 9.16

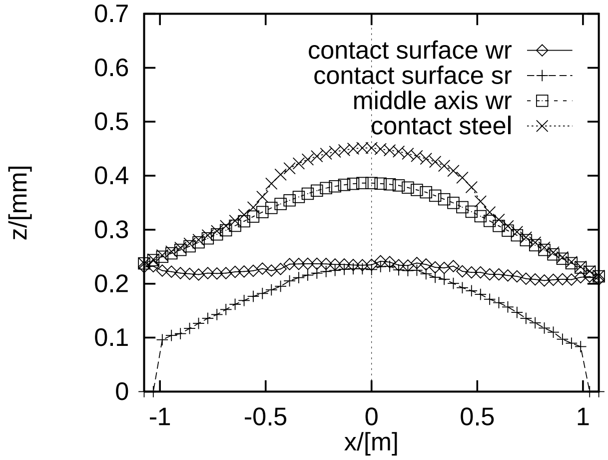


Figure 9.15: sag for $F_{\text{steel}} = 15$KN and $L = 1m$

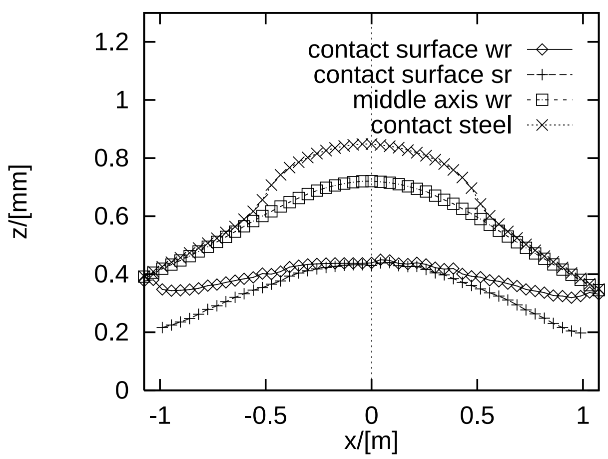


Figure 9.16: sag for $F_{\text{steel}} = 30$KN and $L = 1m$

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Die eitle Einbildung, man verstehe alles, kann ja nur daher kommen, daß man nie etwas verstanden hat. Denn wer nur ein einziges Mal das Verständnis einer Sache erlebt hat, wer wirklich geschmeckt hat, wie man zum Wissen gelangt, der weiß auch, daß er von der Unendlichkeit der übrigen Wahrheiten nichts weiß.

Galileo Galilei

.....nun steh' ich hier ich, armer Tor
und bin so klug wie jeh zuvor.....

Johann Wolfgang von Goethe