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Structure-Acoustic Coupling

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Abstract

The goal of this bachelor thesis is the coupling of the defining equations for mechanics and acoustics to form a so-called structure-acoustic coupling problem. In the first chapter we will therefore develop a model for the mechanical structure by in a first step deriving the three fundamental equations for mechanics (the *equilibrium equation*, the *strain-displacement-relation* and the *constitutive equation*). We will then combine these to arrive at the *Navier-Lamé equation* for isotropic, homogeneous, elastic materials. At the end of the first chapter we consider two special cases (the *plane strain state* and the *plain stress state*). The second chapter deals with a model for acoustics where we derive the corresponding fundamental equations (the *continuity equation*, the *Euler equation* and the *state equation*). We then consider linear versions of these equations and combine them to form the *linear acoustic wave equation*. Then we have a look at two special cases (*plane waves* and *spherical waves*) and at the end of the chapter we derive a *non-linear acoustic wave equation*, also called *Kuznetsov equation*. In the third chapter we will then consider the coupling of both linear models. For this we examine a solid immersed in an acoustic field and find *coupling conditions* which have to hold at the *solid-fluid interface*. Then we transform the Navier-Lamé equation and the linear acoustic wave equation into their *weak formulation* and finally combine them to one system of coupled equations. The last chapter provides some conclusions and gives an outlook on possible further work.

Zusammenfassung

Das Ziel dieser Bachelorarbeit ist die Kopplung der beschreibenden Gleichungen der Mechanik und der Akustik, die dann ein sogenanntes *Fluid-Struktur Kopplungsproblem* bilden. Im ersten Kapitel entwickeln wir daher ein Modell für die Mechanik, angefangen von der Herleitung der drei fundamentalen Gleichungen der Mechanik (die *Gleichgewichtsgleichung*, die *Verzerrungs-Verschiebungs-Relation* und die *Zustandsgleichung*). Diese werden wir dann kombinieren um zur *Navier-Lamé Gleichung* für isotrope, homogene, elastische Materialen zu kommen. Am Ende des ersten Kapitels betrachten wir zwei Spezialfälle (den *ebenen Verzerrungszustand* und den *ebenen Spannungszustand*). Das zweite Kapitel beschäftigt sich mit einem Model für die Akustik, wo wir die entsprechenden beschreibenden Gleichungen herleiten (die *Kontinuitätsgleichung*, die *Bewegungsgleichung* und die *Zustandsgleichung der Akustik*). Wir betrachten dann lineare Versionen dieser Gleichungen und kombinieren sie zur *linearen akustischen Wellengleichung*. Dann folgen zwei Spezialfälle (*Ebene Wellen* und *Kugelwellen*) und am Ende des Kapitels leiten wir eine *nichtlineare akustische Wellengleichung*, auch *Kuznetsov Gleichung* genannt, her. Im dritten Kapitel betrachten wir dann die Kopplung beider linearen Modelle. Dafür analysieren wir einen von einem akustischen Feld umgebenen Festkörper und finden *Kopplungsbedingungen*, welche an der *Schnittstelle zwischen Festkörper und Fluid* gelten müssen. Dann führen wir die Navier-Lamé Gleichung und die lineare akustische Wellengleichung in ihre *schwache Formulierung* über und kombinieren sie zu einem einzigen gekoppelten System von Gleichungen. Im letzten Kapitel finden sich einige Schlussbemerkungen und ein Ausblick auf Möglichkeiten der Weiterarbeit.

Notation

$\mathbb{R}$	set of real numbers
$\boldsymbol{v}$	vector
$\boldsymbol{e}^{(i)}$	i-th Cartesian unit vector
$\boldsymbol{n}$	unit normal vector
Σ_1	set of unit vectors, i.e. $\Sigma_1 \coloneqq \{\boldsymbol{v} \in \mathbb{R}^3 \colon \|\boldsymbol{v}\| = 1\}$
$\partial\Omega$	boundary of Ω
$\bar{\Omega}$	closure of Ω
div	divergence
∇	gradient
curl	curl
Δ	Laplace operator
$\partial/\partial x$	partial derivative
$\partial/\partial \boldsymbol{n}$	normal derivative
d/dx	material/total derivative
$u_{i,j}$	$\frac{\partial u_i}{\partial x_j}$
$\int_\Omega \, \mathrm{dx}$	volume integral
$\int_\Gamma \, \mathrm{ds}_x$	surface integral
$\|\cdot\|$	Euclidean norm
$\langle\cdot,\cdot\rangle$	Euclidean inner product
$A : B$	$\sum_{i,j=1}^{3} A_{ij}B_{ij}$

Contents

Chapter 1

Derivation of a model for the mechanical structure

In this chapter we will develop a model for the mechanical field. Therefore we will derive the three defining equations

- the equilibrium equation,
- the strain-displacement-relation,
- the constitutive equation,

and combine them to arrive at the Navier-Lamé equation. We will then have a look at two special cases, see Section 1.5.

1.1 Stress State - Kinetic

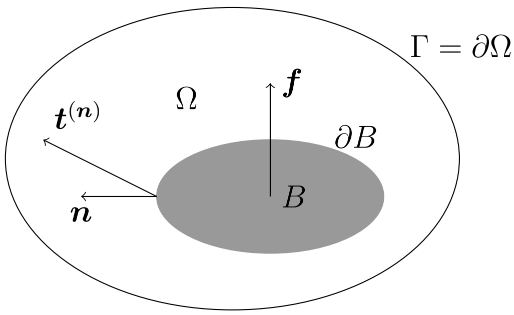

Figure 1.1.1: Mechanical Field

Let us consider a solid body, which occupies the region $\Omega \subset \mathbb{R}^3$ with sufficiently smooth boundary $\Gamma = \partial\Omega$ (see Figure 1.1.1). A bounded sub-region of Ω with sufficiently smooth boundary is called *a part*. We distinguish two kinds of forces acting on Ω:

- *volume forces*: act on all particles of Ω, e.g. gravity or electromagnetic forces.
- *surface tractions*: describe the interaction of two adjacent parts of the body across the separating surface. Surface forces only act on the surface of a body or part of it and do not only depend on the position $\boldsymbol{x} \in \Omega$, but also on the normal vector $\boldsymbol{n}$ on the surface. For example the pressure acting on a solid immersed in water.

It is assumed as an axiom, that a density $\boldsymbol{f}$ (*volume force density*), depending only on the position x, for the total amount of volume forces $\boldsymbol{F}_V(B)$ acting on a part $B \subseteq \Omega$ exists, i.e.

$$\boldsymbol{F}_V(B) = \int_B \boldsymbol{f}(x)\,\mathrm{dx}.$$

Moreover, it is postulated that for all parts B of Ω there exists a density $\boldsymbol{t}$ (*surface traction density*), depending on the position $x \in \partial B$ and the normal vector $\boldsymbol{n}(x)$, for the sum of all surface tractions $\boldsymbol{F}_S(B)$ acting on B, i.e.

$$\boldsymbol{F}_S(B) = \int_{\partial B} \boldsymbol{t}^{(\boldsymbol{n})}(x)\,\mathrm{ds}_x.$$

If $B \neq \Omega$ we can imagine this as *internal surface tractions*. Then $\boldsymbol{t}^{(\boldsymbol{n})}(x)$ describes the force density at which the parts of the body at location x press against each other in direction $\boldsymbol{n}$ (see Figure 1.1.1). Thus, we have that the sum of all forces acting on a part $B \subseteq \Omega$ is given by

$$\boldsymbol{F}(B) = \boldsymbol{F}_V(B) + \boldsymbol{F}_S(B).$$

For a point $x \in \bar{\Omega}$ we define the *stress state* by

$$\{\boldsymbol{t}^{(\boldsymbol{n})}(x) \colon \boldsymbol{n} \in \Sigma_1\}.$$

1.1.1 Cauchy Stress Tensor

If we now fix a point $x \in \Omega$ and consider the surface force density $\boldsymbol{t}$ with respect to the Cartesian coordinate unit vectors $\{\boldsymbol{e}^{(1)}, \boldsymbol{e}^{(2)}, \boldsymbol{e}^{(3)}\}$ as directions, we obtain

$$\begin{aligned}
\boldsymbol{t}^{(\mathbf{e}^{(1)})}(x) &=: (\sigma_{11}(x), \sigma_{12}(x), \sigma_{13}(x))^\top, \\
\boldsymbol{t}^{(\mathbf{e}^{(2)})}(x) &=: (\sigma_{21}(x), \sigma_{22}(x), \sigma_{23}(x))^\top, \\
\boldsymbol{t}^{(\mathbf{e}^{(3)})}(x) &=: (\sigma_{31}(x), \sigma_{32}(x), \sigma_{33}(x))^\top
\end{aligned}$$

the so called *Cauchy stress tensor*

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(x) = [\sigma_{ij}]_{i,j=1}^3,$$

where the (i, j)-th component denotes the j-th component of the force acting on a cutting plane orthogonal to the x_i axis. We call $(\sigma_{11}, \sigma_{22}, \sigma_{33})$ *normal stresses* and $(\sigma_{12}, \sigma_{13}, \sigma_{21}, \sigma_{23}, \sigma_{31}, \sigma_{32})$ *shear stresses.*

Cauchy's stress theorem

Cauchy's stress theorem tells us that it is sufficient to know the Cauchy stress tensor to describe the whole stress state in a point x. It states

$$\boldsymbol{t}_i^{(\boldsymbol{n})}(x) = \sum_{j=1}^{3} \sigma_{ji}(x) n_j(x) = (\boldsymbol{\sigma}^\top \boldsymbol{n})_i \quad \text{for } \boldsymbol{n} = (n_1, n_2, n_3)^\top \in \Sigma_1.$$

The theorem can be proven by considering a body with an oblique cutting face and investigating on the equilibrium of the forces acting on it. For a detailed proof, see [1, p. 52].

1.1.2 Equilibrium of Forces and Moments

We know from Newton's second law of motion that the sum of all forces acting on a body is equal to the mass of the body times its acceleration, which is called inertia. It also states that the sum of all moments is equal to the moment of inertia.

Static Case

In the static case, i.e. a body at rest, there is no acceleration, which means that the sum of all forces acting on all parts of the body has to be zero. This is expressed as

$$\boldsymbol{F}(\Omega') = \boldsymbol{F}_V(\Omega') + \boldsymbol{F}_S(\Omega') = \int_{\Omega'} \boldsymbol{f}(x)\,\mathrm{dx} + \int_{\partial\Omega'} \boldsymbol{t}^{(\boldsymbol{n})}(x)\,\mathrm{ds}_x = \boldsymbol{0}$$

for all $\Omega' \subset \Omega$ with $\partial\Omega'$ sufficiently smooth.
Using Cauchy's stress theorem and Gauß's theorem (see [4, p. 729]) we obtain

$$\int_{\Omega'} \left[\boldsymbol{f}(x) + \operatorname{div}(\boldsymbol{\sigma})\right]\,\mathrm{dx} = \boldsymbol{0}.$$

We now utilize the so-called Euler trick to get rid of the integral.

Theorem 1.1. *(Euler trick) Let $\boldsymbol{f} \in C(\Omega)$. If*

$$\int_V \boldsymbol{f}(x)\,dx = 0$$

for arbitrary $V \subseteq \Omega$, then

$$\boldsymbol{f}(x) = \boldsymbol{0} \quad \forall x \in \Omega.$$

Proof. Assume $\boldsymbol{f}(x) \neq 0$ for some $\bar{x} \in \Omega$. Since $\boldsymbol{f}$ is continuous, it is also non-zero in a neighborhood $U_{\bar{x}}$ of $\bar{x}$. However, choosing $V = U_{\bar{x}}$ yields

$$\int_{U_{\bar{x}}} \boldsymbol{f}(x) \,\mathrm{dx} > 0,$$

which leads to a contradiction. So $\boldsymbol{f}(x) = \mathbf{0}$ for all $x \in \Omega$. $\square$

Since Ω' was chosen arbitrary we may use Theorem 1.1 to drop the integral and arrive at the *equilibrium equation for a body at rest*

$$-\mathrm{div}(\boldsymbol{\sigma}) = \boldsymbol{f}. \tag{1.1}$$

Now we consider the sum of all moments, which should be zero, as for the equilibrium of forces

$$\int_{\Omega'} \boldsymbol{x} \times \boldsymbol{f}(x) \,\mathrm{dx} + \int_{\partial\Omega'} \boldsymbol{x} \times \boldsymbol{t}^{(\boldsymbol{n})}(x) \,\mathrm{ds}_x = 0. \tag{1.2}$$

From this it follows that the stress tensor $\boldsymbol{\sigma}$ is symmetric.

Proof. To keep the proof short we define

$$\sigma_{ji,j} := \sum_{j=1}^{3} \frac{\partial \sigma_{ji}}{\partial x_j} \quad \text{for } i = 1, 2, 3.$$

Then, by using partial integration, it holds that

$$\begin{aligned} \int_{\Omega'} \boldsymbol{x} \times \mathrm{div}\boldsymbol{\sigma} \,\mathrm{dx} &= \int_{\Omega'} \begin{bmatrix} x_2\sigma_{j3,j} - x_3\sigma_{j2,j} \\ x_3\sigma_{j1,j} - x_1\sigma_{j3,j} \\ x_1\sigma_{j2,j} - x_2\sigma_{j1,j} \end{bmatrix} \mathrm{dx} \\ &= \int_{\Omega'} \begin{bmatrix} \sigma_{23} - \sigma_{32} \\ \sigma_{31} - \sigma_{13} \\ \sigma_{12} - \sigma_{21} \end{bmatrix} \mathrm{dx} + \int_{\partial\Omega'} \boldsymbol{x} \times \underbrace{\boldsymbol{\sigma}^\top \boldsymbol{n}}_{=\boldsymbol{t}^{(\boldsymbol{n})}(x)} \,\mathrm{ds}_x. \end{aligned}$$

Substituting this into the equilibrium of moments (1.2) and using (1.1) we get

$$\int_{\Omega'} \boldsymbol{x} \times \underbrace{[\boldsymbol{f} + \mathrm{div}\boldsymbol{\sigma}]}_{=0} - \int_{\Omega'} \begin{bmatrix} \sigma_{23} - \sigma_{32} \\ \sigma_{31} - \sigma_{13} \\ \sigma_{12} - \sigma_{21} \end{bmatrix} \mathrm{dx} = 0.$$

Since Ω' was chosen arbitrary, there must hold $\boldsymbol{\sigma} = \boldsymbol{\sigma}^\top$. $\square$

If we specify a given surface traction $\boldsymbol{g}$ acting on the boundary Γ of Ω this corresponds to the boundary condition

$$\boldsymbol{t}^{(\boldsymbol{n})} = \boldsymbol{\sigma}^\top \boldsymbol{n} = \boldsymbol{g}.$$

Dynamic Case

In the dynamic case we consider a time frame $(0, T)$ and a body Ω with density $\rho(x)$. Let $\boldsymbol{u}(x,t)$ denote the displacement of the body and $\boldsymbol{a}(x,t) = \frac{\partial^2 \boldsymbol{u}}{\partial t^2}(x,t)$ its acceleration. Then we obtain the equation for the equilibrium of forces

$$\int_{\Omega'} \boldsymbol{f}(x,t)\,\mathrm{dx} + \int_{\partial\Omega'} \boldsymbol{t}^{(\boldsymbol{n})}(x,t)\,\mathrm{ds}_x = \int_{\Omega'} \boldsymbol{a}(x,t)\rho(x)\,\mathrm{dx}$$

for all $\Omega' \subset \Omega$ and all $t \in (0,T)$, which leads to the *equilibrium equation for a body in motion*

$$\rho \boldsymbol{a} - \operatorname{div}(\boldsymbol{\sigma}) = \boldsymbol{f} \tag{1.3}$$

by applying the same procedure as in the static case. For the equilibrium of moments we get

$$\int_{\Omega'} \boldsymbol{x} \times \boldsymbol{f}(x,t)\,\mathrm{dx} + \int_{\partial\Omega'} \boldsymbol{x} \times \boldsymbol{t}^{(\boldsymbol{n})}(x,t)\,\mathrm{ds}_x = \int_{\Omega'} \boldsymbol{x} \times \boldsymbol{a}(x,t)\rho(x)\,\mathrm{dx}$$

from which it follows in a similar manner as for (1.2) that the stress tensor is symmetric.

1.2 Strain State - Kinematic

1.2.1 Green-St.Venant Strain Tensor

Under the influence of external forces a material point $\boldsymbol{x}$ of a deformable body Ω moves in a new position $\boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x})$ with the displacement vector

$$\boldsymbol{u}(\boldsymbol{x}) = \begin{pmatrix} u_1(\boldsymbol{x}) \\ u_2(\boldsymbol{x}) \\ u_3(\boldsymbol{x}) \end{pmatrix} \quad \text{for } \boldsymbol{x} \in \bar{\Omega} \subset \mathbb{R}^3.$$

The distance ds between the point $\boldsymbol{x}$ and a neighboring point $\boldsymbol{x} + \boldsymbol{dx}$ changes through deformation to ds', i.e.

$$ds = \|\boldsymbol{dx}\| \quad \text{and} \quad ds' = \|\boldsymbol{dx} + \boldsymbol{du}\|$$

where $\boldsymbol{du} = \boldsymbol{u}(x + \boldsymbol{dx}) - \boldsymbol{u}(x)$. We have

$$(ds')^2 = \langle \boldsymbol{dx} + \boldsymbol{du}, \boldsymbol{dx} + \boldsymbol{du} \rangle = (ds)^2 + 2\langle \boldsymbol{dx}, \boldsymbol{du} \rangle + \|\boldsymbol{du}\|^2.$$

Since we are interested in the case $\boldsymbol{dx} \to \boldsymbol{0}$, the following relation is reasonable

$$\boldsymbol{du} \approx (\nabla \boldsymbol{u})\boldsymbol{dx}.$$

We will now calculate the change in length $(ds')^2 - (ds)^2$

$$\begin{aligned}(ds')^2 - (ds)^2 &= 2\langle \boldsymbol{dx}, \boldsymbol{du} \rangle + \|\boldsymbol{du}\|^2 = 2\langle \boldsymbol{dx}, (\nabla \boldsymbol{u})\boldsymbol{dx} \rangle + \langle (\nabla \boldsymbol{u})\boldsymbol{dx}, (\nabla \boldsymbol{u})\boldsymbol{dx} \rangle \\ &= 2\langle (\nabla \boldsymbol{u})^\top \boldsymbol{dx}, \boldsymbol{dx} \rangle + \langle (\nabla \boldsymbol{u})^\top (\nabla \boldsymbol{u})\boldsymbol{dx}, \boldsymbol{dx} \rangle.\end{aligned}$$

Since

$$2\langle(\nabla \boldsymbol{u})^\top \boldsymbol{dx}, \boldsymbol{dx}\rangle = \langle(\nabla \boldsymbol{u})^\top \boldsymbol{dx}, \boldsymbol{dx}\rangle + \langle(\nabla \boldsymbol{u})\boldsymbol{dx}, \boldsymbol{dx}\rangle$$

we arrive at

$$(ds')^2 - (ds)^2 = 2\langle\frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^\top + \nabla \boldsymbol{u}^\top \nabla \boldsymbol{u})\boldsymbol{dx}, \boldsymbol{dx}\rangle = 2\langle \boldsymbol{e}(\boldsymbol{u})\boldsymbol{dx}, \boldsymbol{dx}\rangle$$

where

$$\boldsymbol{e} = \boldsymbol{e}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^\top + \nabla \boldsymbol{u}^\top \nabla \boldsymbol{u}) \tag{1.4}$$

denotes the *Green-St.Venant strain tensor.* The component e_{kk} describes the relative change in length of a line element ds parallel to the x_k axis for $k = 1, 2, 3$. Whereas e_{kl} for $l \neq k$ describes the angular change between two line elements dx_k and dx_l.

1.2.2 Chauchy's Strain Tensor

If we assume that

$$|(\nabla \boldsymbol{u})_{kl}| \ll 1 \quad \text{for } k = 1, 2, 3,$$

we can neglect the quadratic terms of the Green-St.Venant strain tensor and arrive at the *Cauchy strain tensor* $\boldsymbol{\varepsilon}$

$$\boldsymbol{e}(\boldsymbol{u}) \approx \boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^\top). \tag{1.5}$$

1.3 Constitutive Equation - Hook's Law

For linear elastic material we assume that a linear stress-strain-relation between $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ is given by

$$\sigma_{ij} = \sum_{k,l=1}^{3} D_{ijkl}\, \varepsilon_{kl} \quad i, j = 1, 2, 3 \qquad \text{(Hook's Law)}$$

with the *elastic coefficients* D_{ijkl} for $i, j, k, l = 1, 2, 3$. We call the material homogeneous if the elastic coefficients do not depend on x and in-homogeneous otherwise. From the symmetry of the stress and the strain tensor it follows that the elastic coefficients fulfill

$$D_{ijkl} = D_{klji} \quad \text{and} \quad D_{ijkl} = D_{jikl} = D_{jilk}.$$

From this it follows that we can associate this fourth-order tensor $\boldsymbol{D}$, indexed by (i, j, k, l), with a second-order tensor $\tilde{\boldsymbol{D}}$, indexed by (I, K), in the following way

ij/kl	I/K
11	1
22	2
33	3
23	4
13	5
12	6

and $\tilde{\boldsymbol{D}}$ is symmetric. Thus only 21 of the 81 coefficients of $\boldsymbol{D}$ can be chosen independently. Materials which have no preferred direction, i.e. one can arbitrarily rotate the object and the output of the deformation stays the same, are called *isotropic*. One can show, that for isotropic materials only 2 coefficients can be chosen independently and that for linear elastic isotropic material Hook's law then reads

$$\boldsymbol{\sigma} = \lambda \operatorname{tr}(\boldsymbol{\varepsilon}(\boldsymbol{u}))\boldsymbol{I} + 2\mu \, \boldsymbol{\varepsilon}(\boldsymbol{u}) \tag{1.6}$$

with the so-called *Lamé parameters* λ and μ. For details see [3]. Instead of the Lamé parameters one can use the elasticity modulus E and the Poisson ratio ν which are given by

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \quad \nu = \frac{1}{2}\frac{\lambda}{\lambda + \mu}$$

with the inverse relation

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}.$$

The parameters E and ν can be determined experimentally.

1.4 Navier - Lamé's Equation

1.4.1 Static Case

From

- The kinetic - equilibrium equation (1.1) $-\mathrm{div}\boldsymbol{\sigma} = \boldsymbol{f}$
- The kinematic - strain-displacement-relation (1.5): $\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^\top)$
- The constitutive equation - Hook's law (1.6): $\boldsymbol{\sigma} = \lambda \operatorname{tr}(\boldsymbol{\varepsilon}(\boldsymbol{u}))\boldsymbol{I} + 2\mu \, \boldsymbol{\varepsilon}(\boldsymbol{u})$

we can now derive the *Lamé Equation* for isotropic and homogeneous elastic materials in the static case:

$$\begin{aligned}
\boldsymbol{f} &\overset{(1.1)}{=} -\operatorname{div}\boldsymbol{\sigma} \\
&\overset{(1.6)}{=} -\operatorname{div}(\lambda \operatorname{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}) \\
&\overset{(1.5)}{=} -\lambda \operatorname{div}(\operatorname{tr}(\nabla \boldsymbol{u}))\boldsymbol{I} - \mu \operatorname{div}(\nabla \boldsymbol{u}) - \mu \operatorname{div}(\nabla \boldsymbol{u}^\top) \\
&= -\mu\Delta\boldsymbol{u} - (\lambda+\mu)\nabla(\operatorname{div}\boldsymbol{u}).
\end{aligned}$$

So the Lamé equation now reads

$$-\mu\Delta\boldsymbol{u} - (\lambda+\mu)\nabla(\operatorname{div}\boldsymbol{u}) = \boldsymbol{f}. \tag{1.7}$$

1.4.2 Dynamic Case

For the dynamic case we derived the equilibrium equations (1.3):

$$\rho\frac{\partial^2\boldsymbol{u}}{\partial t^2} - \operatorname{div}\boldsymbol{\sigma} = \boldsymbol{f}.$$

Following the same procedure as above we arrive at the *Navier-Lamé equation* for isotropic and homogeneous elastic materials in the dynamic case:

$$\rho\frac{\partial^2\boldsymbol{u}}{\partial t^2} - \mu\,\Delta\boldsymbol{u} - (\lambda+\mu)\nabla(\operatorname{div}\boldsymbol{u}) = \boldsymbol{f}. \tag{1.8}$$

1.5 Special Cases

1.5.1 Plane Strain State

Assume a body $\mathcal{K} \subset \mathbb{R}^3$ with constant cross-section $\Omega \subset \mathbb{R}^2$, which is much longer in one dimension than in the other two:

$$\mathcal{K} = \{x \in \mathbb{R}^3 \colon (x_1, x_2) \in \Omega, -L < x_3 < L\} \text{ with } L \gg \operatorname{diam}(\Omega).$$

The volume forces $\boldsymbol{f}$ and surface tractions $\boldsymbol{t}$ act on the plane which is orthogonal to the x_3-axis and are independent of x_3, i.e.

$$\boldsymbol{f}(x) = \begin{pmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \\ 0 \end{pmatrix}, \quad \boldsymbol{t}(x) = \begin{pmatrix} t_1(x_1,x_2) \\ t_2(x_1,x_2) \\ 0 \end{pmatrix} \perp x_3.$$

and we assume

$$\varepsilon_{3i} = \varepsilon_{i3} = 0 \text{ for } i = 1,2,3.$$

Then the displacement field $\boldsymbol{u}$ has the following form

$$\boldsymbol{u}(x) = \begin{pmatrix} u_1(x_1, x_2) \\ u_2(x_1, x_2) \\ 0 \end{pmatrix},$$

since

$$0 = \varepsilon_{33} = \frac{\partial u_3}{\partial x_3} \quad \text{and} \quad 0 = -\frac{\partial u_i}{\partial x_3} = \frac{\partial u_3}{\partial x_i} \quad \text{for } i = 1, 2$$

and therefore u_3 is constant. W.l.o.g. we can set this constant to 0. It follows from the assumptions that

$$\sigma_{13} = \sigma_{31} = \sigma_{23} = \sigma_{32} = 0$$

and that σ_{33} can be expressed by σ_{11} and σ_{22}, because

$$\sigma_{33} = \lambda(\varepsilon_{11} + \varepsilon_{22}) = \frac{\lambda}{2(\lambda + \mu)}(\sigma_{11} + \sigma_{22}).$$

It can therefore be eliminated and the Lamé equation now simplifies from a 3D equation to a 2D equation:

$$-\mu\Delta\tilde{\boldsymbol{u}} - (\lambda + \mu)\nabla(\operatorname{div}\, \tilde{\boldsymbol{u}}) = \tilde{\boldsymbol{f}} \text{ with } \tilde{\boldsymbol{u}} = \begin{pmatrix} u_1(x_1, x_2) \\ u_2(x_1, x_2) \end{pmatrix}, \tilde{\boldsymbol{f}} = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}.$$

1.5.2 Plane Stress State

Consider a deformation problem for a plate

$$\mathcal{K} = \{x \in \mathbb{R}^3 \colon (x_1, x_2) \in \Omega, -h < x_3 < h\} \text{ with } h \ll \operatorname{diam}(\Omega).$$

under the influence of x_3-independent volume forces and surface tractions

$$\boldsymbol{f}(x) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \\ 0 \end{pmatrix}, (x_1, x_2) \in \Omega, \quad \boldsymbol{t}(x) = \begin{pmatrix} t_1(x_1, x_2) \\ t_2(x_1, x_2) \\ 0 \end{pmatrix}, (x_1, x_2) \in \Gamma_t$$

and

$$\boldsymbol{t}(x_1, x_2, +h) = \boldsymbol{t}(x_1, x_2, -h) = 0, (x_1, x_2) \in \Omega.$$

It is then assumed that

$$\sigma_{ij}(x_1, x_2, x_3) = \sigma_{ij}(x_1, x_2) \text{ for } i, j = 1, 2$$
$$\sigma_{3i} = \sigma_{i3} = 0 \text{ for } i = 1, 2, 3.$$

From which it follows that ε_{33} can be expressed by ε_{11} and ε_{22} as

$$\varepsilon_{33} = -\frac{\lambda(\varepsilon_{11} + \varepsilon_{22})}{\lambda + 2\mu}.$$

This leads to

$$\bar{\boldsymbol{\sigma}} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \text{ with } \bar{\sigma}_{ij} = \frac{2\lambda\mu}{\lambda + 2\mu}(\varepsilon_{11} + \varepsilon_{22})\delta_{ij} + 2\mu\varepsilon_{ij} \text{ for } i, j = 1, 2$$

and therefore, we arrive arrive at the same equations as for the plane strain state, but with $\bar{\lambda} = \frac{2\lambda\mu}{\lambda+2\mu}$.

Chapter 2

Derivation of a model for the acoustic field

In this chapter we will develop a model for the acoustic field. Therefore we will derive the three defining equations

- the continuity equation,
- Euler's equation,
- the state equation,

and combine them to arrive at the linear acoustic wave equation. We will then consider some acoustic quantities and two special cases and end the chapter with a section on the nonlinear acoustic wave theory.

2.1 Introduction to Acoustics

We will limit ourselves to wave propagation in fluids like air and water, so called non-viscous media. Sound (waves) can be described as small pressure fluctuations in the media. Therefore we introduce the following quantities, which one can compose into their mean part, denoted by a subscript $_0$, and their alternating part, denoted by a prime:

- the density $\rho = \rho_0 + \rho'$,
- the pressure $p = p_0 + p'$,
- the velocity $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{v}'$,

where ρ' is called *acoustic density*, p' the *acoustic pressure* and $\boldsymbol{v}'$ the *acoustic particle velocity*. In the linear theory we assume small fluctuations, i.e. that the alternating parts are much smaller then the mean quantities. We will now derive the three defining equations for the acoustic field.

2.2 Mass Conservation - Continuity Equation

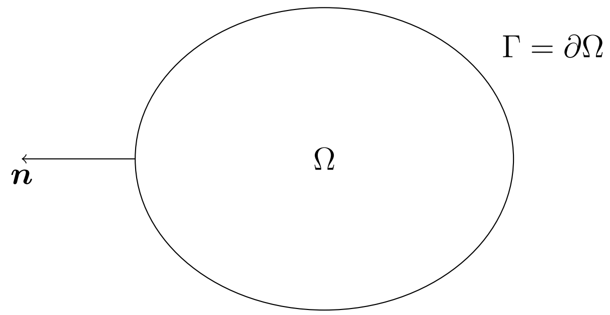

Figure 2.2.1: Acoustic Field

Consider a volume Ω with boundary Γ and given density $\rho(x, t)$ (see Figure 2.2.1). Then the mass m at time t is given by

$$m(t) = \int_\Omega \rho(x, t)\,\mathrm{dx}.$$

If we now consider a time step Δt, a reduction of mass has to be balanced by a mass flow through the surface Γ, which can be stated as

$$\int_\Gamma \rho\,\boldsymbol{v} \cdot \boldsymbol{n}\,\Delta t\,\mathrm{ds}_x = -\int_\Omega (\rho(x, t + \Delta t) - \rho(x, t))\,\mathrm{dx}.$$

Now we multiply the above equation by $\frac{1}{\Delta t}$, take the limit $\Delta t \to 0$ and apply Gauß's theorem (see [4, p. 729]) to obtain

$$\int_\Omega \mathrm{div}(\boldsymbol{v}\rho)\,\mathrm{dx} = -\int_\Omega \frac{\partial \rho}{\partial t}\,\mathrm{dx}.$$

And since Ω was chosen arbitrary we may use Theorem 1.1 and drop the integral to obtain the *continuity equation*

$$\mathrm{div}(\boldsymbol{v}\rho) = -\frac{\partial \rho}{\partial t}. \tag{2.1}$$

2.3 Conservation of Momentum - Euler's Equation

As for the mechanical field we now consider the equilibrium of moments. Newton's first law of motion tells us that force is equal to mass times acceleration. We consider a point $\boldsymbol{x} = (x_1, x_2, x_3)^\top \in \Omega$ and a small cube $\Delta\boldsymbol{x}$ around $\boldsymbol{x}$, i.e.

$$\Delta\boldsymbol{x} = (x_1 - \frac{\Delta x_1}{2}, x_1 + \frac{\Delta x_1}{2}) \times (x_2 - \frac{\Delta x_2}{2}, x_2 + \frac{\Delta x_2}{2}) \times (x_3 - \frac{\Delta x_3}{2}, x_3 + \frac{\Delta x_3}{2}),$$

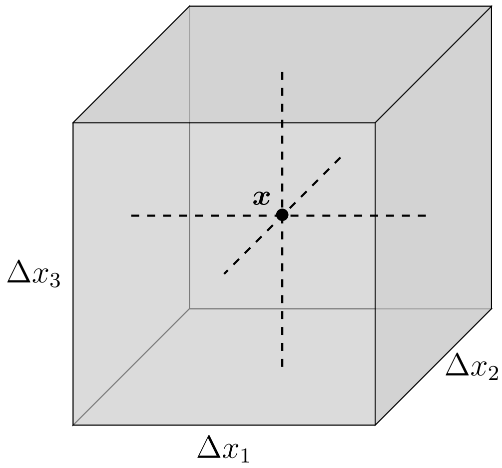

Figure 2.3.1: Cube $\Delta \boldsymbol{x}$ around $\boldsymbol{x} \in \Omega$

with $\Delta x_i \in \mathbb{R}^+$ for $i = 1, 2, 3$, see Figure 2.3.1. The cube's mass is then given by

$$\Delta m = \rho \, \Delta x_1 \, \Delta x_2 \, \Delta x_3.$$

Let $\boldsymbol{a}$ denote the acceleration, which is given by the material derivative (see [1, p. 142]) of the velocity $\boldsymbol{v}$, i.e.

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}, \tag{2.2}$$

since the position of particles is time dependent. Then Newton's law tells us that

$$\Delta \boldsymbol{f} = \boldsymbol{a} \, \Delta m.$$

In our case the force $\Delta \boldsymbol{f}$ is the pressure change in the media which is characterized as $(-p\boldsymbol{n})$ where $\boldsymbol{n}$ denotes the outer normal vector. So we have

$$\begin{aligned}
-\Delta f_1 &= \big(p(x_1 + \frac{\Delta x_1}{2}, x_2, x_3) - p(x_1 - \frac{\Delta x_1}{2}, x_2, x_3)\big) \Delta x_2 \, \Delta x_3, \\
-\Delta f_2 &= \big(p(x_1, x_2 + \frac{\Delta x_2}{2}, x_3) - p(x_1, x_2 - \frac{\Delta x_2}{2}, x_3)\big) \Delta x_1 \, \Delta x_3, \\
-\Delta f_3 &= \big(p(x_1, x_2, x_3 + \frac{\Delta x_3}{2}) - p(x_1, x_2, x_3 - \frac{\Delta x_3}{2})\big) \Delta x_1 \, \Delta x_2,
\end{aligned}$$

and since

$$\begin{aligned}
\big(p(x_1 + \frac{\Delta x_1}{2}, x_2, x_3) - p(x_1 - \frac{\Delta x_1}{2}, x_2, x_3)\big) &\approx \frac{\partial p}{\partial x_1} \Delta x_1, \\
\big(p(x_1, x_2 + \frac{\Delta x_2}{2}, x_3) - p(x_1, x_2 - \frac{\Delta x_2}{2}, x_3)\big) &\approx \frac{\partial p}{\partial x_2} \Delta x_2, \\
\big(p(x_1, x_2, x_3 + \frac{\Delta x_3}{2}) - p(x_1, x_2, x_3 - \frac{\Delta x_3}{2})\big) &\approx \frac{\partial p}{\partial x_3} \Delta x_3,
\end{aligned}$$

we obtain

$$-\Delta \boldsymbol{f} = \nabla p \, \Delta x_1 \, \Delta x_2 \, \Delta x_3. \tag{2.3}$$

Combining (2.2) and (2.3) we now arrive at *Euler's equation*

$$\rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} \right) = -\nabla p. \tag{2.4}$$

2.4 Pressure-Density-Relation - State Equation

We now want to derive a relation between the pressure p and the density ρ. Therefore we consider a fluid in rest, occupying the volume Ω_0 with pressure p_0 and density ρ_0, and a fluid where an acoustic wave propagates, characterized by a volume Ω, pressure p and density ρ. If we now assume no thermal exchange between fluid particles it holds that

$$\left(\frac{\Omega_0}{\Omega} \right)^\kappa = \frac{p}{p_0},$$

where κ denotes the so called adiabatic exponent, for details see [1, p. 143]. If we further assume constant mass, i.e.

$$m_0 = \rho_0 \Omega_0 = \rho \Omega,$$

we get that

$$\left(\frac{\rho}{\rho_0} \right)^\kappa = \frac{p}{p_0}. \tag{2.5}$$

We now consider two types of fluids:

2.4.1 Ideal gases

Let us consider an ideal gas, let T denote the temperature and R the specific gas constant, then we have the relation

$$p = \rho R T$$

and the speed of sound c can be written as

$$c = \sqrt{\kappa R T},$$

see [6, p. 16]. Since we consider the linear acoustic theory we may approximate

$$\left(\frac{\rho}{\rho_0} \right)^\kappa = \left(\frac{\rho_0 + \rho'}{\rho_0} \right)^\kappa \approx 1 + \kappa \frac{\rho'}{\rho_0}$$

and arrive at the *linear state equation for acoustic waves*

$$\frac{p'}{\rho'} = \kappa \frac{p_0}{\rho_0} = c^2. \tag{2.6}$$

2.4.2 Liquids

For liquids we view the pressure p as a quantity depending on the density ρ and the specific entropy s, which is related to heat transfer (see [1, p. 143]). The main pressure density relation is then given by

$$\frac{\partial p(\rho, s)}{\partial \rho} = \frac{K_s}{\rho},$$

where K_s denotes the adiabatic bulk modulus, see [6, p. 16]. Additionally the speed of sound c can now be expressed as

$$c = \sqrt{\frac{K_s}{\rho}},$$

see [6, p. 17].

2.5 Linear Acoustic Wave Equation

In this section we want to combine the three derived equations (2.1), (2.4) and (2.6) to arrive at the linear acoustic wave equation. Since we only consider linear theory for now, we can assume that

$$\boldsymbol{v} \cdot \nabla \rho \ll \rho \,\mathrm{div}\boldsymbol{v} \approx \rho_0 \,\mathrm{div}\boldsymbol{v} \quad \text{and} \quad \boldsymbol{v} \cdot \nabla \boldsymbol{v} \ll \frac{\partial \boldsymbol{v}}{\partial t}.$$

Therefore Euler's equation simplifies to

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\frac{1}{\rho_0} \nabla p \tag{2.7}$$

and the continuity equation to

$$\mathrm{div}\boldsymbol{v} = -\frac{1}{\rho_0} \frac{\partial \rho}{\partial t}. \tag{2.8}$$

Now we may replace the total physical quantities $(\boldsymbol{v}, p, \rho)$ with the acoustic quantities $(\boldsymbol{v}', p', \rho')$, since all derivatives of the mean quantities $(\boldsymbol{v}_0, p_0, \rho_0)$ are zero. By differentiating the linear continuity equation (2.8) with respect to t, we get

$$\frac{\partial}{\partial t} \mathrm{div}\boldsymbol{v}' = -\frac{1}{\rho_0} \frac{\partial^2 \rho'}{\partial t^2},$$

and by switching the order of differentiation the equation becomes

$$\mathrm{div}\left(\frac{\partial \boldsymbol{v}'}{\partial t}\right) = -\frac{1}{\rho_0} \frac{\partial^2 \rho'}{\partial t^2}.$$

Using the linear Euler equation (2.7) yields

$$\operatorname{div}(-\frac{1}{\rho_0}\nabla p') = -\frac{1}{\rho_0}\frac{\partial^2 \rho'}{\partial t^2}. \tag{2.9}$$

Now we combine (2.9) with the linear state equation (2.6) $\rho' = p'/c^2$ to arrive at the *linear acoustic wave equation for* p'

$$\Delta p' = \frac{1}{c^2}\frac{\partial^2 p'}{\partial t^2}. \tag{2.10}$$

One can show that for the acoustic particle velocity $\boldsymbol{v}'$ it holds that $\operatorname{curl}(\boldsymbol{v}') = 0$, i.e. $\boldsymbol{v}'$ is irrotational, see [1, p. 145], and therefore we can write $\boldsymbol{v}'$ as the gradient of a scalar function ψ, which is then called the *acoustic velocity potential*

$$\boldsymbol{v}' = -\nabla\psi. \tag{2.11}$$

It follows from substituting the acoustic velocity potential into (2.7), that the relation between p' and ψ is given by

$$p' = \rho_0\frac{\partial \psi}{\partial t}. \tag{2.12}$$

Thus, we can state the *wave equation for the acoustic velocity potential*

$$\Delta\psi = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}. \tag{2.13}$$

2.6 Acoustic Quantities

In this section we want to state some important acoustic quantities, namely the acoustic energy density, the acoustic energy flux and acoustic impedance. Starting from the linear Euler equation (2.7), taking the dot product with $\boldsymbol{v}'$ and combining this with the linear continuity equation (2.8) and the linear state equation (2.6) yields the following equation

$$\frac{\partial w_a}{\partial t} + \operatorname{div}\boldsymbol{I}_a = 0 \tag{2.14}$$

with the *acoustic energy flux*

$$\boldsymbol{I}_a = \boldsymbol{v}'p'$$

and the *total acoustic energy density*

$$w_a = w_a^{\text{kin}} + w_a^{\text{pot}} \quad \text{with} \quad w_a^{\text{kin}} = \frac{1}{2}\rho_0\|\boldsymbol{v}'\|^2 \quad \text{and} \quad w_a^{\text{pot}} = \frac{(p')^2}{2\rho_0 c^2},$$

which consists of the *acoustic kinetic energy* w_a^{kin} and the *acoustic potential energy density* w_a^{pot}. Let us now consider an acoustic source with constant angular frequency $\omega = 2\pi f$. Then we can write

$$p'(t) = \hat{p}\cos(\omega t) \quad \text{and} \quad \boldsymbol{v}'(t) = \hat{\boldsymbol{v}}\cos(\omega t + \phi).$$

In doing so we can calculate the time averaged acoustic energy density w_a^{av} and intensity $\boldsymbol{I}_a^{av}$ (for details see [1, p. 147]) and get

$$w_a^{av} = \frac{1}{4}\rho_0 \|\hat{\boldsymbol{v}}\|^2 \quad \text{and} \quad \boldsymbol{I}_a^{av} = \frac{\hat{p}\hat{\boldsymbol{v}}}{2}\cos\phi.$$

Replacing the acoustic quantities with their time averaged ones in (2.14) results in

$$\mathrm{div}\boldsymbol{I}_a^{av} = 0$$

which implies

$$\int_\Omega \mathrm{div}\boldsymbol{I}_a^{av}\,\mathrm{dx} = \int_\Gamma \boldsymbol{I}_a^{av}\cdot\boldsymbol{n}\,\mathrm{ds}_x = 0.$$

This only holds if the volume under consideration does not contain any acoustic sources. For the case of a volume which encloses an acoustic source, the result of the integral is the average acoustic power radiated by all enclosed sources P_a^{av}

$$\int_\Gamma \boldsymbol{I}_a^{av}\cdot\boldsymbol{n}\,\mathrm{ds}_x = P_a^{av}.$$

Another important quantity by which we characterize acoustic media (see Section 2.7) is the *acoustic impedance* Z_a

$$Z_a = \frac{p'}{\|\boldsymbol{v}'\|}.$$

2.7 Special Cases

There are two special cases we will treat in this section. First we will look at so-called *plane waves* and then we will have a look at so-called *spherical waves.*

2.7.1 Plane Waves

A wave is called *plane wave* if it shows the same behavior in the plane orthogonal to the direction of propagation (see Figure 2.7.1). This allows us, to write the pressure p as a one-dimensional function in space and the velocity proportional to $\boldsymbol{e}^{(1)}$, i.e.

$$p' = p(x,t) \quad \text{and} \quad \boldsymbol{v}' = v'(x,t)\boldsymbol{e}^{(1)}.$$

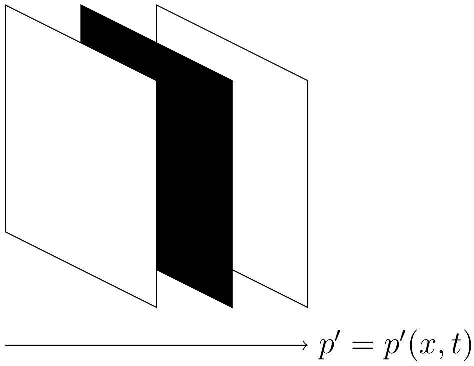

Figure 2.7.1: Plane Wave

By substituting this ansatz into the linear wave equation (2.10), we get

$$\frac{\partial^2 p'}{\partial x^2} - \frac{1}{c}\frac{\partial^2 p'}{\partial t^2} = 0,$$

which could also be written as

$$(\frac{\partial}{\partial x} - \frac{1}{c}\frac{\partial}{\partial t})(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t})p' = 0.$$

By introducing the new variables ξ and η as

$$\xi = t - \frac{x}{c} \quad \text{and} \quad \eta = t + \frac{x}{c},$$

and expressing space and time derivatives by the new variables

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \quad \text{and} \quad \frac{\partial}{\partial x} = \frac{1}{c}(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}), \tag{2.15}$$

the wave equation becomes

$$-\frac{1}{4c^2}\frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta}p' = 0.$$

This means that the unknown p' can be written as the sum of one function depending only on η and one only depending on ξ, i.e.

$$p' = f(\eta) + g(\xi) = f(t - \frac{x}{c}) + g(t + \frac{x}{c}),$$

which describes a wave moving with the speed of sound in direction $+x$ and $-x$. To obtain a relation between p' and v' we utilize the linear state equation (2.6) and use (2.15) in the linear, 1D version of the equations for momentum (2.7) and mass (2.8) conservation, which leads to

$$\frac{\partial}{\partial \xi}(p' - \rho_0 c v') + \frac{\partial}{\partial \eta}(p' + \rho_0 c v') = 0 \quad \text{and} \quad \frac{\partial}{\partial \xi}(\rho_0 c v' - p') + \frac{\partial}{\partial \eta}(\rho_0 c v' + p') = 0.$$

Adding these two equations yields

$$\frac{\partial}{\partial \eta}(p' + \rho_0 c v') = 0$$

which means, that $(p' + \rho_0 c v')$ is a function of ξ, while subtracting the two equations tells us that

$$\frac{\partial}{\partial \xi}(\rho_0 c v' - p') = 0$$

which means that $(\rho_0 c v' - p')$ is a function of η. Thus we have that

$$v' = \frac{1}{\rho_0 c}(f(t - \frac{x}{c}) + g(t + \frac{x}{c})) = \frac{p'}{\rho_0 c}.$$

So for plane waves we have that the ratio $\frac{p'}{v'}$ is constant. This is then called the *specific acoustic impedance*

$$Z_0 = \frac{p'}{v'} = \rho_0 c.$$

If we further look at the acoustic energy density w_a, which simplifies to

$$w_a = \frac{\rho_0 (v')^2}{2} + \frac{(p')^2}{2\rho_0 c^2} = \frac{(p')^2}{\rho_0 c^2},$$

we see that for a plane wave the acoustic kinetic energy and the acoustic potential energy are equal. Considering the acoustic energy intensity $\boldsymbol{I}_a$ we get

$$\boldsymbol{I}_a = \frac{(p')^2}{\rho_0 c}\boldsymbol{e}_1 = c w_a \boldsymbol{e}_1$$

which can be interpreted as the acoustic energy moving with the speed of sound.

2.7.2 Spherical Waves

For a *spherical wave* we assume a point source at the origin and that all quantities involved are independent of the angle ϕ. We now switch to spherical coordinates and rewrite the linear wave equation (2.10). For this we need the laplacian of $p' = p'(r, t)$, which is given by

$$\Delta p'(r, t) = \frac{\partial^2 p'}{\partial r^2} + \frac{2}{r}\frac{\partial p'}{\partial r} = \frac{1}{r}\frac{\partial^2 (r p')}{\partial r^2}.$$

So the wave equation becomes

$$\frac{1}{r}\frac{\partial^2 (r p')}{\partial r^2} - \frac{1}{c^2}\frac{\partial^2 p'}{\partial t^2} = 0. \tag{2.16}$$

If we now express

$$\frac{\partial^2 p'}{\partial t^2} = \frac{1}{r}\frac{\partial^2 (rp')}{\partial t^2}$$

and multiply (2.16) with r, we arrive at the same result as for a plane wave, but with rp' as the unknown function. From Section 2.7.1 we know that the solution is then given by

$$p'(r,t) = \frac{1}{r}\left(f(t-\frac{r}{c}) + g(t+\frac{r}{c})\right),$$

which implies that the pressure amplitude decreases with the distance from the origin. Furthermore the averaged acoustic energy intensity now only depends on the radius and is given by

$$I_r^{av} := \boldsymbol{I}_a^{av} \cdot \boldsymbol{n} = \frac{P^{av}}{4\pi r^2}.$$

This is also called the *spherical spreading law* and describes that the acoustic intensity decreases with the squared distance. The last thing we want to have is again the relation between p' and $\boldsymbol{v}' = v'(r,t)\boldsymbol{e}_r$. After some computation (see [1, p. 151]) one obtains that

$$\lim_{r\to\infty} v'(r,t) = \frac{p'}{\rho_0 c},$$

so that we asymptotically have the same behavior as for plane waves.

2.8 Non-linear Acoustic Wave Equation

In this last section we want to deal with non-linear wave propagation. We will first find non-linear versions of the three defining equations (2.1), (2.4) and (2.6) and then proceed as before by combining these into one equation, the so called *Kuznetsov's equation*. For this we only consider lossy and compressible fluids and further assume that $\boldsymbol{v} = \boldsymbol{v}'$. The first thing to note is, that the continuity equation (2.1) still holds in the non-linear case. So we just have to replace the Euler equation (2.4) and the state equation (2.6).

2.8.1 Non-linear Euler's Equation

In the non-linear case the equation for momentum conservation is the Navier-Stokes equation for compressible fluids (see [1, p. 156])

$$\rho(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla\boldsymbol{v})) + \nabla p = \mu\Delta\boldsymbol{v} + (\frac{\mu}{3} + \zeta)\nabla(\mathrm{div}\boldsymbol{v}),$$

where ζ denotes the bulk viscosity and μ the shear viscosity. Using some vector identities (which assume our domain of interest to be convex) and the assumption $\boldsymbol{v} = \boldsymbol{v}'$ the equation can be rewritten as

$$\rho\frac{\partial \boldsymbol{v}'}{\partial t} + \frac{\rho}{2}\nabla(\|\boldsymbol{v}'\|^2) + \nabla p = (\frac{4\mu}{3} + \zeta)\Delta\boldsymbol{v}' - (\frac{\mu}{3} + \zeta)\mathrm{curl}\,(\mathrm{curl}(\boldsymbol{v}')) + \rho\boldsymbol{v}' \times \mathrm{curl}\boldsymbol{v}'.$$

We may still assume that $\boldsymbol{v}'$ is irrotational, i.e $\mathrm{curl}\boldsymbol{v}' = 0$, so the last two terms, which are related to acoustic streaming, drop and we arrive at our non-linear Euler equation

$$\rho_0\frac{\partial \boldsymbol{v}'}{\partial t} + \rho'\frac{\partial \boldsymbol{v}'}{\partial t} + \frac{\rho_0}{2}\nabla(\|\boldsymbol{v}'\|^2) + \nabla p' = (\frac{4\mu}{3} + \zeta)\Delta\boldsymbol{v}'. \tag{2.17}$$

2.8.2 Non-linear State Equation

According to [1, p. 157] a non-linear relation between pressure and density is given by

$$\rho' = \frac{p'}{c^2} - \frac{1}{\rho_0 c^4}\frac{B}{2A}(p')^2 - \frac{\kappa}{\rho_0 c^4}(\frac{1}{c_\Omega} - \frac{1}{c_p})\frac{\partial p'}{\partial t}, \tag{2.18}$$

where $\frac{B}{A}$ is the non-linearity parameter, κ the adiabatic exponent, c_Ω and c_p the specific heat capacitance at constant volume respectively constant pressure.

2.8.3 Combination of Equations

To combine the equations we approximate every physical quantity in a second order term by it's linear version which we obtained in the previous sections. So we will utilize

$$\rho' \approx \frac{p'}{c^2} \quad (2.6), \quad \nabla p' \approx -\rho_0\frac{\partial \boldsymbol{v}'}{\partial t} \quad (2.7) \quad \text{and} \quad \mathrm{div}\boldsymbol{v}' \approx -\frac{1}{\rho_0}\frac{\partial \rho'}{\partial t} \quad (2.8).$$

Starting from the continuity equation (2.1) in the form

$$\frac{\partial \rho'}{\partial t} + \rho_0\mathrm{div}\boldsymbol{v}' = -\rho'\mathrm{div}\boldsymbol{v}' - \boldsymbol{v}' \cdot \nabla\rho'$$

these approximations yield

$$\frac{\partial \rho'}{\partial t} + \rho_0\mathrm{div}\boldsymbol{v}' = \frac{1}{2\rho_0 c^4}\frac{\partial (p')^2}{\partial t} + \frac{\rho_0}{2c^2}\frac{\partial \|\boldsymbol{v}'\|^2}{\partial t}.$$

By using the non-linear state equation (2.18) this becomes

$$\begin{aligned}\frac{1}{c^2}\frac{\partial p'}{\partial t} - \frac{1}{\rho_0 c^4}\frac{B}{2A}\frac{\partial (p')^2}{\partial t} - \frac{\kappa}{\rho_0 c^4}(\frac{1}{c_\Omega} - \frac{1}{c_p})\frac{\partial^2 p'}{\partial t^2} + \rho_0\mathrm{div}\boldsymbol{v}' \\ = \frac{1}{2\rho_0 c^4}\frac{\partial (p')^2}{\partial t} + \frac{\rho_0}{2c^2}\frac{\partial \|\boldsymbol{v}'\|^2}{\partial t}.\end{aligned} \tag{2.19}$$

If we now use a similar procedure for the non-linear Euler equation (see [1, p. 158]) we get

$$\rho_0 \frac{\partial \boldsymbol{v}'}{\partial t} + \nabla p' = \frac{1}{2\rho_0 c^2}\nabla (p')^2 - \frac{\rho_0}{2}\nabla \|\boldsymbol{v}'\|^2 - \frac{1}{\rho_0 c^2}(\frac{4\mu}{3} + \zeta)\nabla \frac{\partial p'}{\partial t}. \qquad (2.20)$$

Just as in the linear theory (Section 2.5) we differentiate (2.19) with respect to t and apply the divergence operator to (2.20). Then we subtract the resulting equations, interchange the order of differentiation and use the linear relations between space and time derivatives

$$\Delta p' = \frac{1}{c^2}\frac{\partial^2 p'}{\partial t^2} \quad \text{and} \quad \Delta \boldsymbol{v}' = \frac{1}{c^2}\frac{\partial^2 \boldsymbol{v}'}{\partial t^2},$$

to finally arrive at the non-linear wave equation for p'

$$\Delta p' - \frac{1}{c^2}\frac{\partial^2 p'}{\partial t^2} = -\frac{b}{c^2}\frac{\partial \Delta p'}{\partial t} - \frac{1}{\rho_0 c^4}\frac{B}{2A}\frac{\partial^2 p'}{\partial t^2} - \frac{\rho_0}{c^2}\frac{\partial^2 \|\boldsymbol{v}'\|^2}{\partial t^2},$$

where

$$b = \frac{1}{\rho_0}(\frac{4\mu}{3} + \zeta) + \frac{\kappa}{\rho_0}(\frac{1}{c_\Omega} - \frac{1}{c_p})$$

denotes the *diffusivity of sound.* As in the linear theory, we can still express $\boldsymbol{v}'$ and p' by the vector potential ψ as

$$\boldsymbol{v}' = -\nabla \psi \quad \text{and} \quad p' = \rho_0 \frac{\partial \psi}{\partial t}.$$

Using this ansatz we have finally derived the so-called *Kuznetsov equation* for non-linear acoustics

$$c^2 \Delta \psi - \frac{\partial^2 \psi}{\partial t^2} = -\frac{\partial}{\partial t}\left(b\Delta\psi + \frac{1}{c^2}\frac{B}{2A}(\frac{\partial \psi}{\partial t})^2 + \|\nabla \psi\|^2 \right).$$

Chapter 3

Coupling of the two models

In the final chapter we consider a solid immersed in an acoustic field, e.g. a loudspeaker and the air surrounding it. Mechanical vibrations in the solid cause acoustic waves in the surrounding which act as surface tractions on the vibrating structure. We will derive the weak formulation of both PDE's and incorporate coupling conditions. When talking about solid fluid interaction there are two scenarios:

1. *Strong coupling:* Here we need to solve both equations and their coupling conditions simultaneously, e.g. a piezoelectric ultrasound array immersed in water.

2. *Weak coupling:* If the pressure of the fluid on the solid can be neglected, we may compute the solutions sequentially, i.e we first calculate the mechanical vibrations and then use them as input for the acoustic field. For example the acoustic noise of an electric transformer.

3.1 Solid-Fluid Interface

To ensure that the velocity field is continuous, we need that at the interface between solid and fluid the normal components of both velocity fields coincide. Let $\boldsymbol{u}$ be the displacement of the mechanical field and denote by $\boldsymbol{v}$ the mechanical velocity, defined as $\boldsymbol{v} = \partial \boldsymbol{u}/\partial t$. Furthermore, let $\boldsymbol{v}'$ be the acoustic particle velocity, which we expressed by the acoustic scalar potential ψ as $\boldsymbol{v}' = -\nabla\psi$, see (2.11). Then we must have that

$$\boldsymbol{n} \cdot (\boldsymbol{v} - \boldsymbol{v}') = 0$$

which means that

$$\boldsymbol{n} \cdot \frac{\partial \boldsymbol{u}}{\partial t} = -\boldsymbol{n} \cdot \nabla\psi = -\frac{\partial\psi}{\partial \boldsymbol{n}}. \tag{3.1}$$

Additionally we have to take in account that the fluid acts as a surface traction on the solid, which can be stated as

$$\boldsymbol{t}^{(\boldsymbol{n})} = -\boldsymbol{n}p' =^{(2.12)} -\boldsymbol{n}\rho_0 \frac{\partial \psi}{\partial t}, \tag{3.2}$$

where p' denotes the acoustic pressure and ρ_0 the mean density of the fluid.

3.2 Coupled formulation

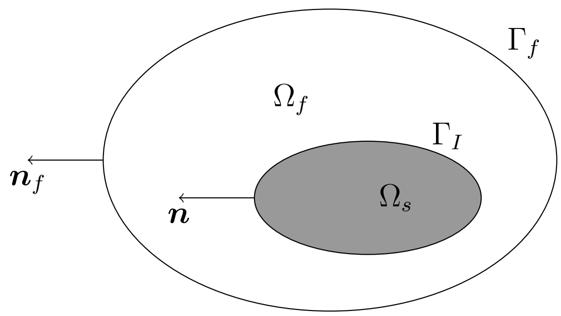

Figure 3.2.1: Coupled Field

We now want to derive the coupled system of equations in its weak formulation. Let Ω_s be the domain for the mechanical field, Ω_f the domain for the acoustic field, Γ_I the interface between these two with normal vector $\boldsymbol{n}$ facing from solid to fluid, and Γ_f the boundary of the fluid, with outer normal vector $\boldsymbol{n}_f$ (see Figure 3.2.1).

3.2.1 Mechanical field

Starting from the equilibrium equation for the dynamic case (1.3)

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} - \mathrm{div}\boldsymbol{\sigma} = \boldsymbol{f}$$

we multiply with a test function $\boldsymbol{\varphi}$ and integrate over Ω_s

$$\int_{\Omega_s} \rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} \cdot \boldsymbol{\varphi} \, \mathrm{dx} - \int_{\Omega_s} \mathrm{div}\boldsymbol{\sigma} \cdot \boldsymbol{\varphi} \, \mathrm{dx} = \int_{\Omega_s} \boldsymbol{f} \cdot \boldsymbol{\varphi} \, \mathrm{dx}.$$

Using integration by parts on the divergence term we obtain

$$\int_{\Omega_s} \rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} \cdot \boldsymbol{\varphi} \, \mathrm{dx} + \int_{\Omega_s} \boldsymbol{\sigma} \colon \nabla \boldsymbol{\varphi} \, \mathrm{dx} - \int_{\Gamma_I} \underbrace{(\boldsymbol{\sigma n})}_{\boldsymbol{t}^{(\boldsymbol{n})}} \cdot \boldsymbol{\varphi} \, \mathrm{ds}_x = \int_{\Omega_s} \boldsymbol{f} \cdot \boldsymbol{\varphi} \, \mathrm{dx}.$$

Since $\boldsymbol{\sigma} = \boldsymbol{\sigma}^\top$ we have

$$\boldsymbol{\sigma} \colon \nabla\boldsymbol{\varphi} = \frac{1}{2}(\boldsymbol{\sigma} \colon \nabla\boldsymbol{\varphi}) + \frac{1}{2}\underbrace{(\boldsymbol{\sigma}^\top \colon \nabla\boldsymbol{\varphi})}_{\boldsymbol{\sigma} \colon (\nabla\boldsymbol{\varphi})^\top} = \boldsymbol{\sigma} \colon \frac{1}{2}(\nabla\boldsymbol{\varphi} + \nabla\boldsymbol{\varphi}^\top) = \boldsymbol{\sigma} \colon \boldsymbol{\varepsilon}(\boldsymbol{\varphi}).$$

From the constitutive law (1.6)

$$\boldsymbol{\sigma} = \lambda \underbrace{\operatorname{tr}(\boldsymbol{\varepsilon}(\boldsymbol{u}))}_{\operatorname{tr}(\nabla\boldsymbol{u}) = \operatorname{div}\boldsymbol{u}} \boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u})$$

we get that

$$\begin{aligned}
\boldsymbol{\sigma} \colon \boldsymbol{\varepsilon}(\boldsymbol{\varphi}) &= (2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda(\operatorname{div}\boldsymbol{u})\boldsymbol{I}) \colon \boldsymbol{\varepsilon}(\boldsymbol{\varphi}) \\
&= 2\mu(\boldsymbol{\varepsilon}(\boldsymbol{u}) \colon \boldsymbol{\varepsilon}(\boldsymbol{\varphi})) + \lambda(\operatorname{div}\boldsymbol{u}) \underbrace{(\boldsymbol{I} \colon \boldsymbol{\varepsilon}(\boldsymbol{\varphi}))}_{\operatorname{tr}(\boldsymbol{\varepsilon}(\boldsymbol{\varphi})) = \operatorname{div}\boldsymbol{\varphi}} \\
&= 2\mu(\boldsymbol{\varepsilon}(\boldsymbol{u}) \colon \boldsymbol{\varepsilon}(\boldsymbol{\varphi})) + \lambda(\operatorname{div}\boldsymbol{u})(\operatorname{div}\boldsymbol{\varphi}).
\end{aligned}$$

So the weak formulation is given by

$$\begin{aligned}
\int_{\Omega_s} \rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} \cdot \boldsymbol{\varphi} \, \mathrm{dx} + \int_{\Omega_s} 2\mu(\boldsymbol{\varepsilon}(\boldsymbol{u}) \colon \boldsymbol{\varepsilon}(\boldsymbol{\varphi})) + \lambda(\operatorname{div}\boldsymbol{u})(\operatorname{div}\boldsymbol{\varphi}) \, \mathrm{dx} \\
- \int_{\Gamma_I} \boldsymbol{t}^{(\boldsymbol{n})} \cdot \boldsymbol{\varphi} \, \mathrm{ds}_x = \int_{\Omega_s} \boldsymbol{f} \cdot \boldsymbol{\varphi} \, \mathrm{dx}.
\end{aligned} \tag{3.3}$$

3.2.2 Acoustic field

For the acoustic field we start with the wave equation for the acoustic velocity potential (2.13)

$$\Delta\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2},$$

multiply it with a test function w and integrate over Ω_f

$$-\int_{\Omega_f} \Delta\psi \, w \, \mathrm{dx} + \int_{\Omega_f} \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} w \, \mathrm{dx} = 0.$$

Using partial integration on the Laplace term we obtain the weak formulation for the acoustic field

$$\int_{\Omega_f} \nabla\psi \cdot \nabla w \, \mathrm{dx} - \int_{\Gamma_f} w \, \boldsymbol{n}_f \cdot \nabla\psi \, \mathrm{ds}_x + \int_{\Gamma_I} w \, \boldsymbol{n} \cdot \nabla\psi \, \mathrm{ds}_x + \int_{\Omega_f} \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} w \, \mathrm{dx} = 0, \tag{3.4}$$

where the plus sign in front of the interface surface integral comes from the choice of the normal vector $\boldsymbol{n}$.

3.2.3 Coupled System

Now we incorporate the coupling conditions, i.e.

$$\int_{\Gamma_I} \boldsymbol{t}^{(\boldsymbol{n})} \cdot \boldsymbol{\varphi} \, \mathrm{ds}_x =^{(3.2)} - \int_{\Gamma_I} p' \boldsymbol{n} \cdot \boldsymbol{\varphi} \, \mathrm{ds}_x = - \int_{\Gamma_I} \rho_0 \frac{\partial \psi}{\partial t} \boldsymbol{n} \cdot \boldsymbol{\varphi} \, \mathrm{ds}_x.$$

For simplicity we assume $\partial \psi / \partial \boldsymbol{n} = 0$ on Γ_f so we just have to deal with

$$\int_{\Gamma_I} w \, \boldsymbol{n} \cdot \nabla \psi \, \mathrm{ds}_x = \int_{\Gamma_I} w \, \frac{\partial \psi}{\partial \boldsymbol{n}} \, \mathrm{ds}_x =^{(3.1)} - \int_{\Gamma_I} w \, \boldsymbol{n} \cdot \frac{\partial \boldsymbol{u}}{\partial t} \, \mathrm{ds}_x.$$

So the final coupled system of equations is given by

$$\begin{aligned} \int_{\Omega_s} \rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} \cdot \boldsymbol{\varphi} \, \mathrm{dx} + \int_{\Omega_s} 2\mu(\boldsymbol{\varepsilon}(\boldsymbol{u}) \colon \boldsymbol{\varepsilon}(\boldsymbol{\varphi})) + \lambda(\mathrm{div} \boldsymbol{u})(\mathrm{div} \boldsymbol{\varphi}) \, \mathrm{dx} & \\ + \int_{\Gamma_I} \rho_0 \frac{\partial \psi}{\partial t} \boldsymbol{n} \cdot \boldsymbol{\varphi} \, \mathrm{ds}_x = \int_{\Omega_s} \boldsymbol{f} \cdot \boldsymbol{\varphi} \, \mathrm{dx} & \\ \int_{\Omega_f} \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} w \, \mathrm{dx} + \int_{\Omega_f} \nabla \psi \cdot \nabla w \, \mathrm{dx} - \int_{\Gamma_I} w \, \boldsymbol{n} \cdot \frac{\partial \boldsymbol{u}}{\partial t} \, \mathrm{ds}_x = 0. & \end{aligned}$$

Chapter 4

Conclusions

In the course of this thesis we managed to derive all the necessary tools to model a structure-acoustic coupling problem. For the mechanical model we started with the equilibrium equation (1.3), derived a linear relation between the displacement and the strain in (1.5) and combined both of them with Hook's law for isotropic, elastic materials (1.6) to arrive at the Navier-Lamé equation (1.8). For the acoustic model we first derived the continuity equation (2.1) and the Euler equation (2.4). Additionally we used the linear state equation (2.6) and linear versions of the Euler (2.7) and the continuity equation (2.8), to arrive at the linear acoustic wave equation (2.10). For the coupled system we derived coupling conditions, (3.1) and (3.2), and deduced the weak formulations for the mechanical (3.3) and the acoustic field (3.4). By incorporating the coupling conditions we reached our final system of coupled equations:

$$
\begin{aligned}
\int_{\Omega_s} \rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} \cdot \boldsymbol{\varphi} \, \mathrm{dx} + \int_{\Omega_s} 2\mu(\boldsymbol{\varepsilon}(\boldsymbol{u}) \colon \boldsymbol{\varepsilon}(\boldsymbol{\varphi})) + \lambda(\mathrm{div}\boldsymbol{u})(\mathrm{div}\boldsymbol{\varphi}) \, \mathrm{dx} \\
+ \int_{\Gamma_I} \rho_0 \frac{\partial \psi}{\partial t} \boldsymbol{n} \cdot \boldsymbol{\varphi} \, \mathrm{ds}_x = \int_{\Omega_s} \boldsymbol{f} \cdot \boldsymbol{\varphi} \, \mathrm{dx} \\
\int_{\Omega_f} \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} w \, \mathrm{dx} + \int_{\Omega_f} \nabla \psi \cdot \nabla w \, \mathrm{dx} - \int_{\Gamma_I} w \, \boldsymbol{n} \cdot \frac{\partial \boldsymbol{u}}{\partial t} \, \mathrm{ds}_x = 0.
\end{aligned}
$$

From this one can start to think about numerical computations. One way is to apply the finite element method to the coupled system of equations, see for example [1]. Other methods have been proposed, see for example [7] or [8]. For a further study in acoustics and structure-acoustic coupling one may refer to [9].

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Ich, Lukas Burgholzer, erkläre an Eides statt, dass ich die vorliegende Bachelorarbeit selbständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.

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