AUTOMATIC GENERATION OF COUPLED FINITE ELEMENT-BOUNDARY ELEMENT DISCRETIZATION

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ABSTRACT: This paper presents an adaptive FEM-BEM coupling method to solve non-linear problems involving elasto-plastic deformations. Unlike the existing coupling approaches, the present method facilitates an automatic generation of the FEM zone/zones of discretization so as to include zones where non-linearity (plasticity) occurs. The generation of the FEM and BEM zones of discretization eliminates the need of a user predefinition of the discretization zones. Furthermore, the FEM and BEM zones of discretization are subsequently adapted according to the state of computation. The presented adaptive FEM-BEM coupling method employs considerably smaller FEM meshes, which plainly leads to reduction of required system resources.

Keywords: FEM; BEM; Adaptive coupling, Elasto-plasticity.

INTRODUCTION

In many practical applications, nonlinearities are concentrated in relatively small portions of the computational domain (plasticity, phase change, etc.). In such case the idea of using the finite element method (FEM) to deal with nonlinearities while using the boundary element method (BEM) in the remaining portion of the body naturally arises. Examples include, but not limited to, the detailed stress analysis in the surrounding of pressure tunnels in an infinite/semi-infinite geological medium. The generalized structure that can have nonlinear characteristics and possibly contains a nonlinear and/or nonhomogeneous part of the surrounding is modeled with the FEM. The remaining unbounded geological medium may be best represented by the BEM. The same is true for many problems in solid mechanics. Typical two-dimensional bodies with their discretization by coupled FEM-BEM are given in Figure 1.

The idea of coupling FEM and BEM was addressed for the first time by Zienkiewicz et al.[1, 2], suggesting a '*mariage à la mode - the best of both worlds*'. The theory and algorithms of coupling FEM and BEM solutions reached, by now, a fairly matured state. In the conventional (direct) FEM-BEM coupling formulation (see, e.g., references[1-12] not to mention many others), the discretized equations for both the FEM and BEM sub-domains are combined, resulting in a global coupled system of equations.

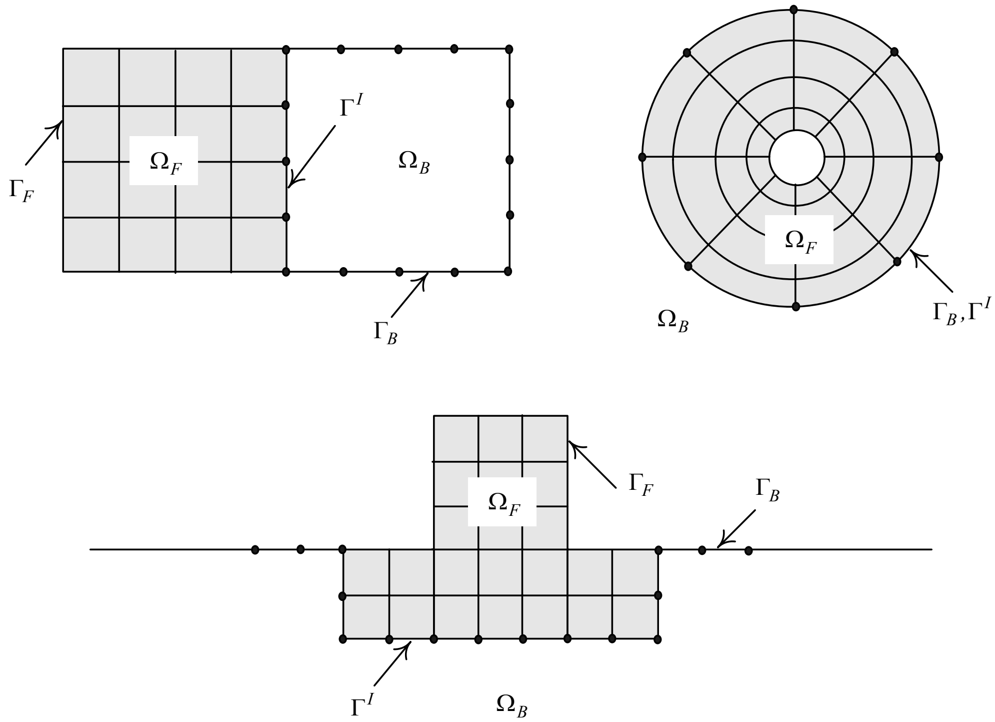

Figure 1. FEM-BEM discretization of two-dimensional bodies

As an alternative to the conventional (direct) FEM-BEM coupling approach, iterative solution procedures can be employed where the equations for both the FEM and BEM sub-domains are solved separately, avoiding the assembly and solution of a global coupled system of equations. Boundary conditions at the coupling interface are updated iteratively until convergence is achieved. In order to achieve and accelerate convergence, relaxation parameters are often applied within the iterative procedure. Interface relaxation FEM-BEM coupling methods are discussed in details in references[13-19].

The symmetric coupling of BEM and FEM goes back to Costabel[3,4]. With the Symmetric Galerkin Boundary Element Method, FEM-like stiffness matrices can be produced which are suitable for coupling the Boundary Element Method and the Finite Element Method (see, e.g., references[5,6,8,11,12,20,21]). During the last decade iterative substructuring solvers for symmetric coupled boundary and finite element equations have been developed by Langer[21], Haase et al.[20], Hsiao et al.[22], Steinbach[23] for elliptic boundary value problems in bounded and unbounded, two and three-dimensional domains. Parallel implementations showed high performance on several platforms[20]. Langer and Steinbach[24] introduced the boundary element tearing and interconnecting (BETI) methods as boundary element counterparts of the well-established finite element tearing and interconnecting (FETI) methods. As a logical continuation of the BETI technique, Langer and Steinbach[25] introduced the coupled finite and boundary element tearing and interconnecting methods (FETI/BETI).

Brink et al.[7] investigated a coupling of mixed finite elements and Galerkin boundary elements in linear elasticity, taking into account adaptive mesh refinement based on a posteriori error estimators. Carstensen et al.[9] presented an *h*-adaptive FEM-BEM coupling algorithm (mesh refinement of the boundary elements and the finite elements) for the solution of viscoplastic and elasto-plastic interface problems. Mund and Stephan[10] derived a posteriori error estimate for nonlinear-coupled FEM-BEM equations by using hierarchical basis techniques. They presented an algorithm for adaptive error control, which allows independent refinements of the finite elements and the boundary elements.

In the existing FEM-BEM coupling approaches, the finite element and the boundary element zones of discretization (FEM and BEM sub-domains) are defined a priori and do not change during the computation. This requires preliminary expert knowledge about the problem at hand and the computational cost can be higher than necessary depending on the definition of the finite element and boundary element zones of discretization (FEM and BEM sub-domains). This paper presents an adaptive FEM-BEM coupling method to solve non-linear problems involving elasto-plastic deformations. The present method facilitates an automatic generation of the FEM zone/zones of discretization so as to include zones where non-linearity (plasticity) occurs. Furthermore, the FEM and BEM zones of discretization are subsequently adapted according to the state of computation. The paper is organized as follows: Section 2 gives a brief overview of the basic BEM equations in elasticity, FEM equations in elasto-plasticity, compatibility and equilibrium conditions at the FEM-BEM sub-domains interface and the interface relaxation methods for coupling of FEM and BEM discretized systems of equations. Section 3 discusses the preliminaries of our coupling method, general features of the adaptive concept and the progressive adaption of the FEM and BEM zones of discretization. Example application in Section 4 shows the potential of our adaptive FEM-BEM coupling method.

BASIC EQUATIONS

The adaptive FEM-BEM coupling method given in succeeding sections is capable of utilizing conventional (direct) or domain decomposition (preconditioning/interface relaxation) approaches for coupling of FEM and BEM discretized systems of equations. The novelty of our adaptive coupling method is the automatic generation and successive adaption of the FEM and BEM zones of discretization.

Let us consider an arbitrary domain, Ω, with known boundary conditions specified at the entire boundary, Γ. If in some portion of Ω, the problem to be solved is non-linear and in remaining part linear or domain itself is infinite, we decompose it into two sub-domains, namely, Ω_F and Ω_B. In all following equations subscripts F and B stand for *Finite Element* and *Boundary Element* sub-domains, respectively. Decomposition yields unknown values of displacements $\mathbf{u}_F^I$, $\mathbf{u}_B^I$, forces $\mathbf{f}_F^I$, and tractions $\mathbf{t}_B^I$ at the interface, *I*, of sub-domains. In the remaining parts of sub-domains, we define displacements as, $\mathbf{u}_F^F$, $\mathbf{u}_B^B$, forces $\mathbf{f}_F^F$, and tractions $\mathbf{t}_B^B$.

Elastic region - BEM basic equations

Disregarding body forces, the assembled boundary element equations, in a partitioned form, are given by:

$$\begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_B^B \\ \mathbf{u}_B^I \end{Bmatrix} = \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{t}_B^B \\ \mathbf{t}_B^I \end{Bmatrix} \text{ on } \mathit{\Gamma}_B \tag{1}$$

where $\mathbf{H}$ and $\mathbf{G}$ denote the influence coefficient matrices.

Elasto-plastic region – FEM basic equations

For an elasto-plastic analysis, the incremental form of the FEM equations, in a partitioned form, can be written as:

$$\begin{Bmatrix} \mathit{\Delta}\boldsymbol{\psi}_F^F \\ \mathit{\Delta}\boldsymbol{\psi}_F^I \end{Bmatrix} = \begin{bmatrix} \mathbf{K}_{T11} & \mathbf{K}_{T12} \\ \mathbf{K}_{T21} & \mathbf{K}_{T22} \end{bmatrix} \begin{Bmatrix} \mathit{\Delta}\mathbf{u}_F^F \\ \mathit{\Delta}\mathbf{u}_F^I \end{Bmatrix} - \begin{Bmatrix} \mathit{\Delta}\mathbf{f}_F^F \\ \mathit{\Delta}\mathbf{f}_F^I \end{Bmatrix} \text{ in } \Omega_F \tag{2}$$

where $\mathbf{K}_T$ is the tangent stiffness matrix and $\Delta\boldsymbol{\psi}$ is the residual (or out-of-balance) force vector. It should be noted that for each load increment Equations (2) are nonlinear and therefore are solved iteratively.

Standard solution procedure, at each load increment, contains iterations over computations of tangent stiffness (based on current stress, and plastic strain, if required), applied loads based on current configuration, internal force and force residual. Then, displacement increment is calculated. With updated displacements the plastic strain increments at element integration points are obtained. Finally, check on convergence is carried out. If the procedure converged, plastic strains are updated and next increment proceeds.

Compatibility and equilibrium conditions

At the sub-domains interface, the compatibility and equilibrium conditions are:

$$\mathbf{u}_B^I = \mathbf{u}_F^I \quad on \ \Gamma^I \tag{3}$$

$$\mathbf{f}_F^I + \mathbf{M}\,\mathbf{t}_B^I = 0 \quad on \ \Gamma^I \tag{4}$$

where $\mathbf{M}$ is the transforming matrix, due to the weighting of the boundary tractions by the interpolation functions at the interface.

Coupled FEM-BEM discretized systems of equations

In this subsection we give a brief description of the interface relaxation coupling approach[13,16]. The Dirichlet-Neumann coupling method in elasto-plasticity is outlined as:

Set initial guess $\left(\mathbf{u}_B^I\right)_{n=0}$. (where n is the iteration number).

For $n = 0, 1, 2, \ldots$, do until convergence

BEM sub-domain:

solve $\begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{bmatrix} \begin{Bmatrix} \left(\mathbf{u}_B^B\right)_n \\ \left(\mathbf{u}_B^I\right)_n \end{Bmatrix} = \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{bmatrix} \begin{Bmatrix} \left(\mathbf{t}_B^B\right)_n \\ \left(\mathbf{t}_B^I\right)_n \end{Bmatrix}$ for $\left(\mathbf{t}_B^I\right)_n$

solve $\left(\mathbf{f}_F^I\right)_n + \mathbf{M}\left(\mathbf{t}_F^I\right)_n = 0$ for $\left(\mathbf{f}_F^I\right)_n$

FEM sub-domain ($i = 1, 2, \ldots, m$, where m is a specified number of increments):

solve $\begin{Bmatrix} \mathit{\Delta}\boldsymbol{\psi}_F^F \\ \mathit{\Delta}\boldsymbol{\psi}_F^I \end{Bmatrix} = \begin{bmatrix} \mathbf{K}_{T11} & \mathbf{K}_{T12} \\ \mathbf{K}_{T21} & \mathbf{K}_{T22} \end{bmatrix} \begin{Bmatrix} \left(\mathit{\Delta}\mathbf{u}_F^F\right)_n \\ \left(\mathit{\Delta}\mathbf{u}_F^I\right)_n \end{Bmatrix}_i - \begin{Bmatrix} \left(\mathit{\Delta}\mathbf{f}_F^F\right)_n \\ \left(\mathit{\Delta}\mathbf{f}_F^I\right)_n \end{Bmatrix}_i$ for $\left(\left(\mathit{\Delta}\mathbf{u}_F^I\right)_n\right)_i$

apply $\left(\left(\mathbf{u}_F^I\right)_n\right)_{i+1} = \left(\left(\mathbf{u}_F^I\right)_n\right)_i + \left(\left(\Delta\mathbf{u}_F^I\right)_n\right)_i$ and get $\left(\mathbf{u}_F^I\right)_n \equiv \left(\left(\mathbf{u}_F^I\right)_n\right)_m$

apply $\left(\mathbf{u}_B^I\right)_{n+1} = (1-\theta)\left(\mathbf{u}_B^I\right)_n - \theta\left(\mathbf{u}_F^I\right)_n$ where θ is a relaxation parameter to ensure and/or accelerate convergence

The geometric contraction based coupling method in elasto-plasticity is outlined as:
Set initial guess $\left(\mathbf{u}_F^I\right)_{n=0}$ and $\left(\mathbf{u}_B^I\right)_{n=0}$ (where n is the iteration number).
For $n = 0, 1, 2, \ldots$, do until convergence

BEM sub-domain:

solve $\begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{bmatrix} \begin{Bmatrix} \left(\mathbf{u}_B^B\right)_n \\ \left(\mathbf{u}_B^I\right)_n \end{Bmatrix} = \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{bmatrix} \begin{Bmatrix} \left(\mathbf{t}_B^B\right)_n \\ \left(\mathbf{t}_B^I\right)_n \end{Bmatrix}$ for $\left(\mathbf{t}_B^I\right)_n$

FEM sub-domain ($i = 1, 2, \ldots, m$, where m is a specified number of increments):

solve $\begin{Bmatrix} \Delta\boldsymbol{\psi}_F^F \\ \Delta\boldsymbol{\psi}_F^I \end{Bmatrix} = \begin{bmatrix} \mathbf{K}_{T11} & \mathbf{K}_{T12} \\ \mathbf{K}_{T21} & \mathbf{K}_{T22} \end{bmatrix} \begin{Bmatrix} \left(\Delta\mathbf{u}_F^F\right)_n \\ \left(\Delta\mathbf{u}_F^I\right)_n \end{Bmatrix}_i - \begin{Bmatrix} \left(\Delta\mathbf{f}_F^F\right)_n \\ \left(\Delta\mathbf{f}_F^I\right)_n \end{Bmatrix}_i$ for $\left(\left(\Delta\mathbf{f}_F^I\right)_n\right)_i$

apply $\left(\left(\mathbf{f}_F^I\right)_n\right)_{i+1} = \left(\left(\mathbf{f}_F^I\right)_n\right)_i + \left(\left(\Delta\mathbf{f}_F^I\right)_n\right)_i$ and get $\left(\mathbf{f}_F^I\right)_n \equiv \left(\left(\mathbf{f}_F^I\right)_n\right)_m$

solve $\left(\mathbf{f}_F^I\right)_n = \mathbf{M}\left(\mathbf{t}_F^I\right)_n$ for $\left(\mathbf{t}_F^I\right)_n$

apply $\left(\mathbf{u}_B^I\right)_{n+1} = \left(\mathbf{u}_B^I\right)_n - \alpha\left(\mathbf{t}_B^I + \mathbf{t}_F^I\right)_n$ where α is a relaxation parameter to ensure and/or accelerate convergence.

apply $\left(\mathbf{u}_F^I\right)_{n+1} = \left(\mathbf{u}_B^I\right)_{n+1}$

ADAPTIVE COUPLING OF FEM AND BEM

As pointed out previously, in the available FEM-BEM coupling approaches, the finite element and the boundary element zones of discretization (FEM and BEM sub-domains) are defined a priori and do not change during the computation. This requires preliminary expert knowledge about the problem at hand. Furthermore, a predefined FEM zone of discretization will probably result in either an under/overestimation of the nonlinear region where the FEM is employed. In the former case, inaccurate solutions is obtained to the problem at hand while for the later the computational cost may be higher than necessary. The key idea of our adaptive coupling method is to eliminate the disadvantages of a prior definition and manual localization of the FEM and BEM zones of discretization. The basic steps of implementation of the adaptive concept in elasto-plasticity are summarized as follows (Figure 2):

1. Data input and initial BEM discretization.
   In our adaptive method, the user has only to define an initial BEM discretization as in the case of elastic analysis.
2. Load increment and BEM elastic analysis with the initial discretization.
   The implementation of the BEM computation is straightforward.
3. Detection of FEM discretization zones.
   The detection of FEM discretization zones can be carried out by a loop over stress/strain based maximum values computed at predefined points inside the BEM sub-domain (within a user-defined distance). In this work, violation to the yield condition is utilized as a discretization measure (predictor) for the generation of FEM zones of discretization.

However, this predictor based on elastic analysis will give a rough, though not accurate, estimate of the zones sensible for discretization by the FEM (Figure 3).

4. Automatic generation of FEM zone of discretization.
   If the chosen discretization criterion is fulfilled at a particular region, the region is replaced by a FEM discretization.
   It is useful to reuse the BEM internal points as finite element nodes for the FEM descritezed region, as they are conveniently distributed in the particular area of interest. This will result in a reduction of the complexity of data management and ease of the automatic generation and adaption of the FEM zone of discretization.
5. Construction of the interface for the coupling of both FEM and BEM discretizations.
   In order to ensure the compatible coupling between the remaining BEM zone and the FEM zone, the interface is constructed reflecting the current situation.
6. Coupled FEM-BEM stress analysis involving elasto-plastic deformations.
   For the present work and without loss of generality, we choose the interface relaxation approach for the coupling of FEM and BEM discretized system of equations.
7. Consequent to the FEM-BEM coupled analysis; the finite elements at the coupling interface are checked for yield condition.
   If any finite element at the coupling interface is found to be yielded, the BEM zone surrounding the yielded element is included into the FEM zone of discretization. This can be achieved by adding finite elements to the nodes at the interface that belong to the yielded element (within a prescribed distance). A coupled FEM-BEM analysis is then performed and the process is iterated (Figure 4).
8. Next load increment requires a repetition of steps 2-7.

EXAMPLE APPLICATION

In this section we present an example application of coupling FEM-BEM by means of the adaptive algorithm presented in Section 3. Calculation results are compared with those obtained by conventional FEM models (with rigid truncation boundary).

The application deals with the stress analysis in a surrounding of a deep tunnel (radius, $r = 1.0\,\mathrm{m}$) subjected to a non-uniform pressure, P, which is assumed to be as high as 70 Mpa (Figure 5). The analysis involves elasto-plastic deformations in geological medium of infinite extent. Material properties of the geological medium are described by Young’s modulus $E = 21\,\mathrm{GPa}$, Poisson's ratio $\nu = 0.18$, cohesion $c = 10$ MPa and angle of internal friction $\phi = 41^o$. Moreover, we assume a Drucker-Prager formulation, with no hardening effect, as a yield function and plane strain loading conditions. Figure 5 further shows the initial BEM discretization. The loads are applied incrementally (total of 7 increments is used). Figure 6 and 7 shows the automatically generated coupled FEM-BEM discretization at increments 2 and 7 (that is equivalent to loads of P=20 MPa and P=70 MPa), respectively. Figures 6 and 7 show the equivalent plastic strains computed using our adaptive FEM-BEM coupling method at increments 2 and 7. The figures further compare the computed results with those obtained by conventional finite element solutions. For the FEM models, the infinite domain is truncated at radius 8.7 and 15 m from the center of the tunnel (448 and 608 quadrilateral finite elements), respectively. The FEM solutions converge to those of the adaptive coupled FEM-BEM method.

The results clearly show the advantages of the adaptive coupled FEM-BEM models in terms of accuracy, no-requirement of a user predefinition of the FEM and BEM discretization zones and smaller FEM meshes.

CONCLUSIONS

This paper presents an adaptive coupled FEM-BEM for elasto-plastic analysis. The FEM zone of discretization is automatically generated. FEM and BEM zones of discretization are progressively adapted according to the state of computation. The key idea of the adaptive FEM-BEM coupling method is the elimination of the disadvantages of prior definition and manual localization of the FEM and BEM zones of discretization. Furthermore, the adaptive FEM-BEM coupling method employs considerably smaller FEM meshes while achieving better accuracy.

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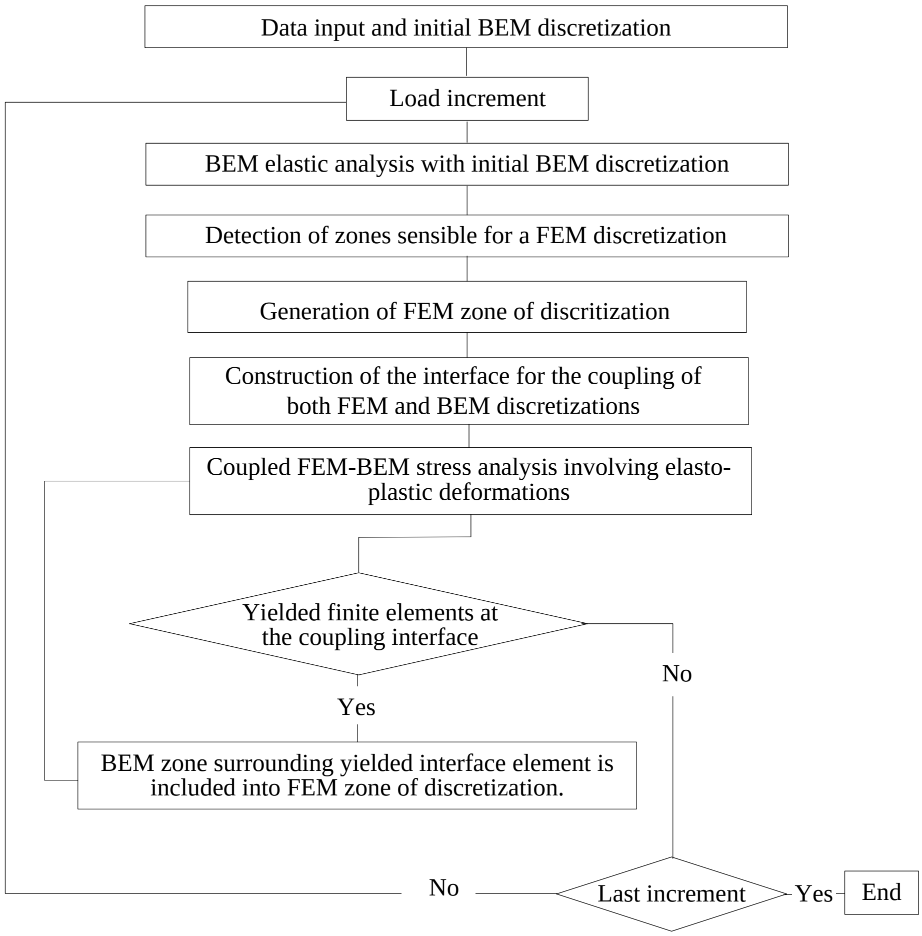

Figure 2: Adaptive FEM-BEM coupling method for elasto-plastic analysis

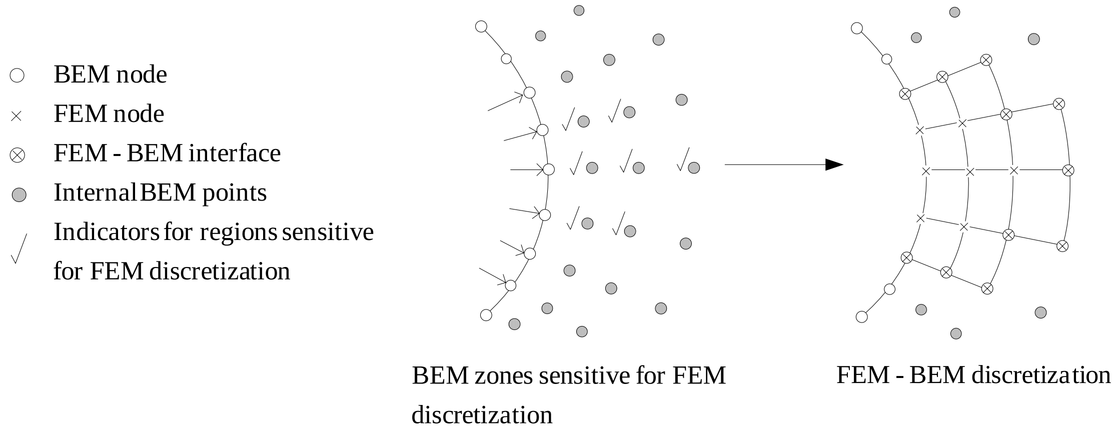

Figure 3: Generation of FEM zone of discretization

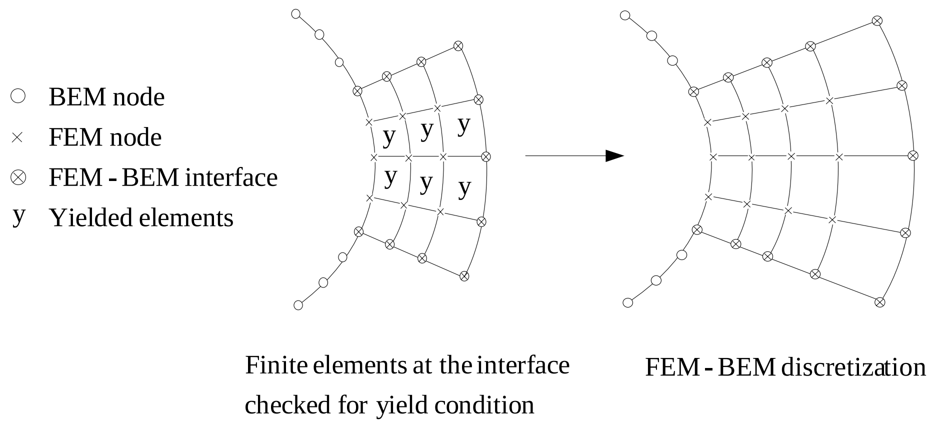

Figure 4: FEM discretization for regions surrounding yielding elements at the coupling interface

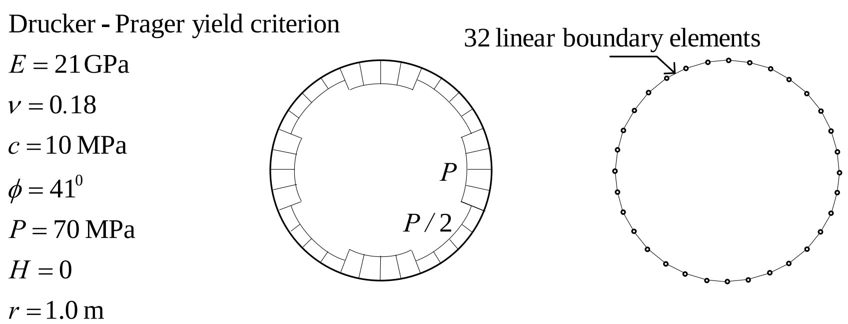

Figure 5: Deep tunnel subjected to non-uniform pressure and initial BEM discretization

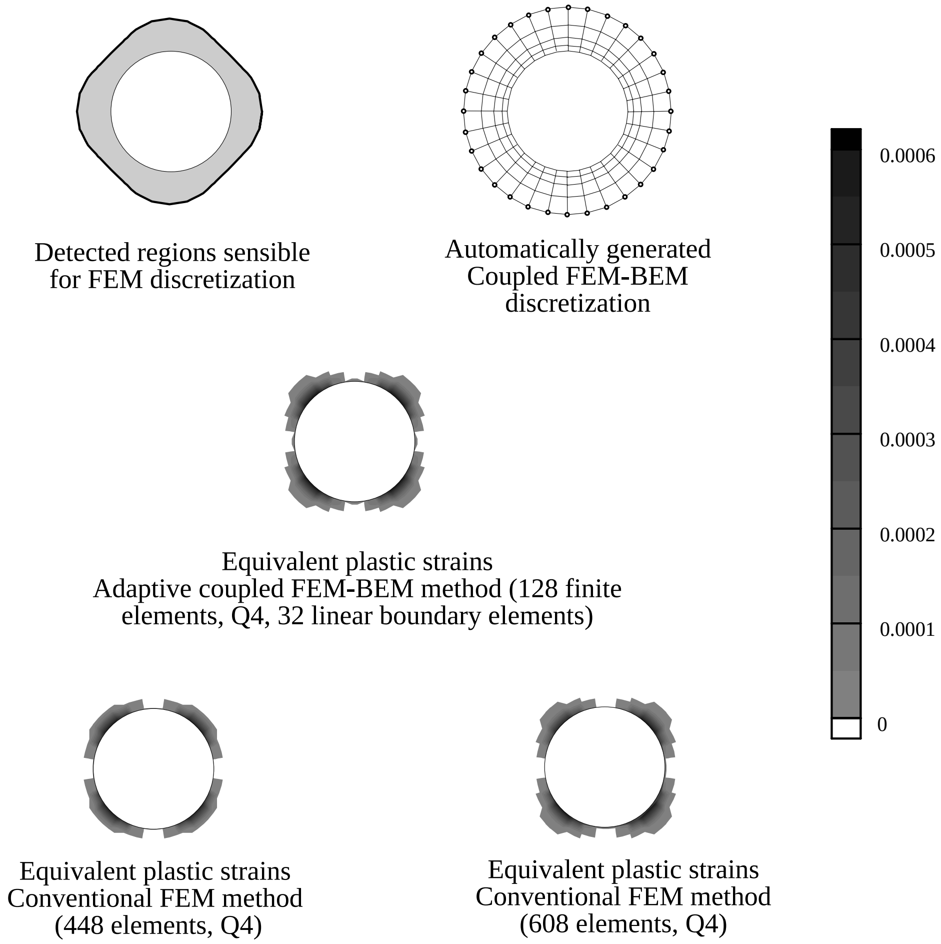

Figure 6: Adaptive coupled FEM-BEM method and comparison of final solutions with conventional FEM – increment2

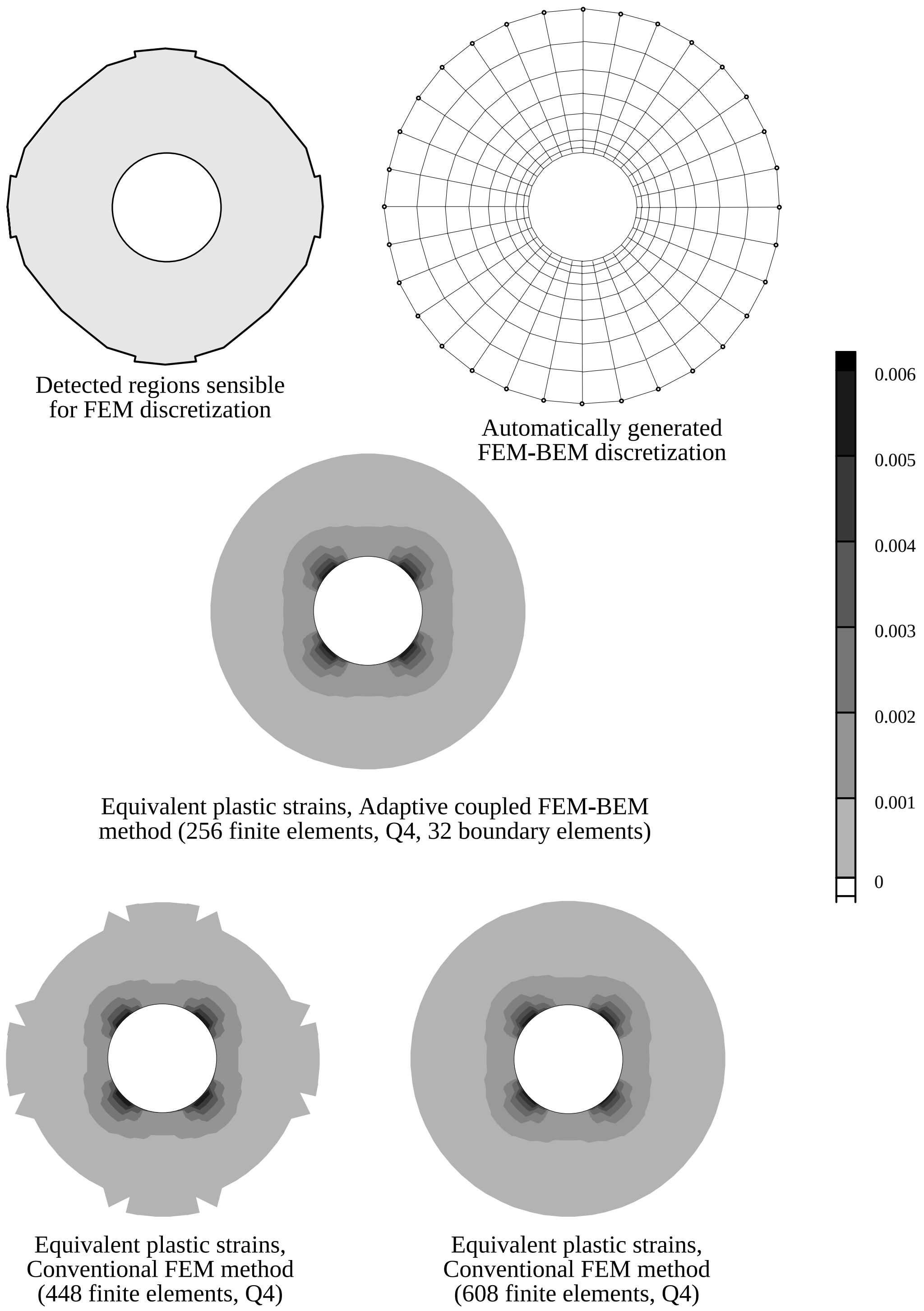

Figure 7: Adaptive coupled FEM-BEM method and comparison of final solutions with conventional FEM – increment7

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