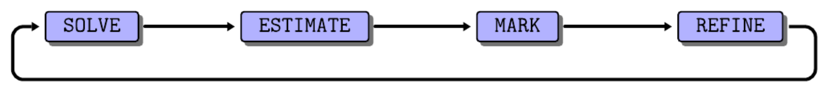
Axioms of adaptivity
Part I

Rainer Schneckenleitner

Institute of Computational Mathematics (NUMA)
Johannes Kepler University Linz (JKU)
Austria

January 7, 2020 - Linz, Austria



Outline

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- $\mathcal{X}$ is a vector space
- $\mathcal{T}_0$ is an initial shape-regular triangulation
- $\mathbb{T} := \{\mathcal{T} : \mathcal{T}$ is an admissible refinement of $\mathcal{T}_0\}$
- $\mathbb{T}(N) := \{\mathcal{T} \in \mathbb{T} : |\mathcal{T}| - |\mathcal{T}_0| \leq N\}$
- Each $T \in \mathcal{T}$ is split into $C_{son} \geq 2$ elements in case of refinement
- Each $\mathcal{T} \in \mathbb{T}$ induces a finite dimensional space $\mathcal{X}(\mathcal{T})$
- There exists a solver

$$U(\cdot) : \mathbb{T} \to \mathcal{X}(\cdot)$$

which provides an approximation $U(\mathcal{T}) \in \mathcal{X}(\mathcal{T})$ to a limit

$$u \in \mathcal{X}$$

- $T \in \mathcal{T}$ admits a refinement indicator

$$\eta_T(\mathcal{T};\cdot) : \mathcal{X}(\mathcal{T}) \to [0,\infty)$$

- The summation of η_T over all $T \in \mathcal{T}$ gives the global error estimator

$$\eta(\mathcal{T};V)^2 := \sum_{T\in\mathcal{T}} \eta_T(\mathcal{T};V)^2$$

for all $V \in \mathcal{X}(\mathcal{T})$

- We assume the existence of an error measure $\mathrm{d}[\mathcal{T};\cdot,\cdot]$ in $\mathcal{X} \cup \mathcal{X}(\mathcal{T})$ which fulfills
  - $\mathrm{d}[\mathcal{T};v,w] \geq 0$,
  - $\mathrm{d}[\mathcal{T};v,w] \leq C_\Delta \mathrm{d}[\mathcal{T};w,v]$
  - $C_\Delta^{-1}\mathrm{d}[\mathcal{T};v,y] \leq \mathrm{d}[\mathcal{T};v,w] + \mathrm{d}[\mathcal{T};w,y]$

for all $v,w,y \in \mathcal{X} \cup \mathcal{X}(\mathcal{T})$, $C_\Delta > 0$

- For a refinement $\widehat{\mathcal{T}} \in \mathbb{T}$ of $\mathcal{T} \in \mathbb{T}$, $\mathrm{d}[\widehat{\mathcal{T}}; \cdot, \cdot]$ is well-defined on $\mathcal{X} \cup \mathcal{X}(\mathcal{T}) \cup \mathcal{X}(\widehat{\mathcal{T}})$ with

$$\mathrm{d}[\widehat{\mathcal{T}}; v, w] = \mathrm{d}[\mathcal{T}; v, w]$$

  for all $v \in \mathcal{X}, w \in \mathcal{X}(\mathcal{T})$
- For each $\varepsilon > 0$ there exists a refinement $\widehat{\mathcal{T}} \in \mathbb{T}$ of $\mathcal{T} \in \mathbb{T}$ such that

$$\mathrm{d}[\widehat{\mathcal{T}}; u, U(\widehat{\mathcal{T}})] \leq \varepsilon$$

- θ is a bulk parameter with

$$0 < \theta \leq 1$$

  $\theta = 1$: uniform refinement

Outline

Adaptive algorithm (Alg. 1)

INPUT: $\mathcal{T}_0$ and θ

For $\ell = 0, 1, 2, \ldots$ do

1. Compute discrete approximation $U(\mathcal{T}_\ell)$
2. Compute refinement indicators $\eta_T(\mathcal{T}_\ell : U(\mathcal{T}_\ell))$ for all $T \in \mathcal{T}_\ell$
3. Determine set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ of (almost) minimal cardinality such that
$$\theta \eta(\mathcal{T}_\ell; U(\mathcal{T}_\ell))^2 \leq \sum_{T \in \mathcal{M}_\ell} \eta_T(\mathcal{T}_\ell; U(\mathcal{T}_\ell))^2$$
4. Refine (at least) the marked elements $T \in \mathcal{M}_\ell$ to generate the triangulation $\mathcal{T}_{\ell+1}$

OUTPUT: Discrete approximations $U(\mathcal{T}_\ell)$ and error estimators $\eta(\mathcal{T}_\ell : U(\mathcal{T}_\ell))$ for all $\ell \in \mathbb{N}_0 := 0, 1, 2, 3, \ldots$

Conditions on the mesh

- The used mesh-refinement strategy has to satisfy

$$|\mathcal{T}\setminus\widehat{\mathcal{T}}| \leq |\widehat{\mathcal{T}}| - |\mathcal{T}|, \tag{1}$$

$$|\mathcal{T}_{\ell+1}| - |\mathcal{T}_\ell| \leq (C_{son} - 1)|\mathcal{T}_\ell| \text{ for all } \ell \in \mathbb{N}, \tag{2}$$

$$|\mathcal{T}_\ell| - |\mathcal{T}_0| \leq C_{mesh} \sum_{k=0}^{\ell-1} |\mathcal{M}_k| \text{ for all } \ell \in \mathbb{N} \tag{3}$$

for all refinements $\widehat{\mathcal{T}} \in \mathbb{T}$ of $\mathcal{T} \in \mathbb{T}$, some $\mathbb{T}-$dependent constant $C_{mesh} > 0$ and for any two meshes $\mathcal{T}, \mathcal{T}' \in \mathbb{T}$ there is a coarsest common refinement $\mathcal{T} \oplus \mathcal{T}' \in \mathbb{T}$ such that

$$|\mathcal{T} \oplus \mathcal{T}'| \leq |\mathcal{T}| + |\mathcal{T}'| - |\mathcal{T}_0| \tag{4}$$

Outline

Set of axioms

- (A1) Stability
- (A2) Reduction property
- (A3) General quasi-orthogonality
- (A4) Discrete reliability

Stability (A1)

Stability on non-refined element domains: For all refinements $\widehat{\mathcal{T}} \in \mathbb{T}$ of a triangulation $\mathcal{T} \in \mathbb{T}$, for all subsets $\mathcal{S} \subseteq \mathcal{T} \cap \widehat{\mathcal{T}}$ of non-refined element domains and for all $V \in \mathcal{X}(\mathcal{T})$, $\widehat{V} \in \mathcal{X}(\widehat{\mathcal{T}})$, it holds that

$$\left| \left(\sum_{T \in \mathcal{S}} \eta_T(\widehat{\mathcal{T}}; \widehat{V})^2 \right)^{1/2} - \left(\sum_{T \in \mathcal{S}} \eta_T(\mathcal{T}; V)^2 \right)^{1/2} \right| \leq C_{stab} \mathrm{d}[\widehat{\mathcal{T}}; \widehat{V}, V] \tag{A1}$$

with a constant $C_{stab} > 0$.

Reduction property (A2)

Reduction property on refined element domains: Any refinement $\widehat{\mathcal{T}} \in \mathbb{T}$ of a triangulation $\mathcal{T} \in \mathbb{T}$ satisfies

$$\sum_{T \in \widehat{\mathcal{T}} \setminus \mathcal{T}} \eta_T(\widehat{\mathcal{T}}; U(\widehat{\mathcal{T}}))^2 \leq \rho_{red} \sum_{T \in \mathcal{T} \setminus \widehat{\mathcal{T}}} \eta_T(\mathcal{T}; U(\mathcal{T}))^2 + C_{red} \mathrm{d}[\widehat{\mathcal{T}}; U(\widehat{\mathcal{T}}), U(\mathcal{T})] \quad \text{(A2)}$$

with $C_{red} > 0$ and $0 < \rho_{red} < 1$.

General quasi-orthogonality (A3)

General quasi-orthogonality: There exist constants

$$0 \leq \varepsilon_{qo} < \varepsilon^*_{qo}(\theta) := \sup_{\delta>0} \frac{1-(1+\delta)(1-(1-\rho_{red})\theta)}{C^2_{rel}(C_{red}+(1+\delta^{-1})C^2_{stab})}$$

and $C_{qo}(\varepsilon_{qo}) \geq 1$ such that the output of Alg. 1 satisfies, for all $\ell, N \in \mathbb{N}_0$ with $N \geq \ell$, that

$$\sum_{k=\ell}^{N} \left(\mathrm{d}[\mathcal{T}_{k+1}; U(\mathcal{T}_{k+1}), U(\mathcal{T}_k)]^2 - \varepsilon_{qo}\mathrm{d}[\mathcal{T}_k; u, U(\mathcal{T}_k)]^2\right) \leq C_{qo}(\varepsilon_{qo})\eta(\mathcal{T}_\ell; U(\mathcal{T}_\ell))^2. \quad \text{(A3)}$$

Discrete reliability (A4)

Discrete reliability: For all refinements $\widehat{\mathcal{T}} \in \mathbb{T}$ of a triangulation $\mathcal{T} \in \mathbb{T}$, there exists a subset $\mathcal{R}(\mathcal{T}, \widehat{\mathcal{T}}) \subseteq \mathcal{T}$ and constants $C_{ref}, C_{drel} > 0$ with $\mathcal{T} \backslash \widehat{\mathcal{T}} \subseteq \mathcal{R}(\mathcal{T}, \widehat{\mathcal{T}})$ and $|\mathcal{R}(\mathcal{T}, \widehat{\mathcal{T}})| \leq C_{ref} |\mathcal{T} \backslash \widehat{\mathcal{T}}|$ such that

$$\mathrm{d}[\widehat{\mathcal{T}}; U(\widehat{\mathcal{T}}), U(\mathcal{T})]^2 \leq C_{drel}^2 \sum_{T \in \mathcal{R}(\mathcal{T}, \widehat{\mathcal{T}})} \eta_T(\mathcal{T}; U(\mathcal{T}))^2. \qquad \text{(A4)}$$

Discrete reliability implies reliability

Lemma

Discrete reliability (A4) *implies reliability in the sense that any triangulation* $\mathcal{T} \in \mathbb{T}$ *satisfies with* $C_{rel} = C_\Delta C_{drel}$

$$\mathrm{d}[\mathcal{T}; u, U(\mathcal{T})] \leq C_{rel}\eta(\mathcal{T}; U(\mathcal{T})). \tag{5}$$

Quasi-monotonicity of the error estimator

Lemma

Stability (A1), *reduction* (A2), *and discrete reliability* (A4) *imply quasi-monotonicity, i.e., there exists a constant* $C_{mon} > 0$ *such that all refinements* $\widehat{\mathcal{T}} \in \mathbb{T}$ *of* $\mathcal{T} \in \mathbb{T}$ *satisfy*

$$\eta(\widehat{\mathcal{T}}; U(\widehat{\mathcal{T}})) \leq C_{mon}\eta(\mathcal{T}; U(\mathcal{T})). \tag{6}$$

Quasi-monotonicity of the error estimator

Lemma

Stability (A1), *reduction* (A2), *reliability* (5) *and a Céa type best approximation, i.e., there is a constant* $C_{Céa} > 0$ *such that*

$$\mathrm{d}[\widehat{\mathcal{T}}; u, U(\widehat{\mathcal{T}})] \leq C_{Céa} \min_{V \in \mathcal{X}(\widehat{\mathcal{T}})} \mathrm{d}[\widehat{\mathcal{T}}; u, V]. \qquad (7)$$

holds for any refinement $\widehat{\mathcal{T}}$ *of* $\mathcal{T} \in \mathbb{T}$*. Suppose that the ansatz spaces* $\mathcal{X}(\mathcal{T}) \subseteq \mathcal{X}(\widehat{\mathcal{T}})$ *are nested. Then, the error estimator is quasi-monotone* (6)*.*

A concrete example

Consider the homogeneous Poisson problem in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d, d \geq 2$, find $u \in \mathcal{X} := H_0^1(\Omega)$ such that

$$a(u, v) := \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \int_\Omega fv \,\mathrm{d}x =: \langle F, v \rangle \qquad \forall v \in \mathcal{X}.$$

Suppose $\mathcal{X}(\mathcal{T}) \subseteq \mathcal{X}(\widehat{\mathcal{T}}) \subseteq \mathcal{X}$ for all triangulations $\mathcal{T} \in \mathbb{T}$ and all refinements $\widehat{\mathcal{T}} \in \mathbb{T}$ of $\mathcal{T}$ and suppose
$\mathrm{d}[\mathcal{T}; v, w] = \mathrm{d}[v, w] = \|v - w\|_{\mathcal{X}}$ for all $\mathcal{T} \in \mathbb{T}$ and $v, w \in \mathcal{X}$.
The Lax-Milgram lemma yields solutions in the continuous as well as in the discrete cases. Furthermore, there holds a priori convergence

$$\lim_{\ell \to \infty} \|u - U(\mathcal{T}_\ell)\|_{\mathcal{X}} = 0.$$

A concrete example

Hence, stability (A1), reduction (A2) and reliability (5) already imply convergence.
Galerkin orthogonality yields

$$a(u - U(\mathcal{T}_{\ell+1}), V) = 0 \qquad \forall V \in \mathcal{X}(\mathcal{T}_{\ell+1})$$

and implies the Pythagoras theorem

$$\|u - U(\mathcal{T}_{\ell+1})\|_{\mathcal{X}}^2 = \|u - U(\mathcal{T}_\ell)\|_{\mathcal{X}}^2 - \|U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell)\|_{\mathcal{X}}^2.$$

General quasi-orthogonality (A3) is already satisfied.
It remains to show stability (A1), reduction (A2) and discrete reliability (A4).

Outline

References

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