Exercise sheet 3

1. To appear.

2. As shown in Lemma 4.4, the p-Laplace operator A is strictly monotone. Assuming two solutions $u, v \in X := H_0^1(\Omega)$, we have by subtracting their weak forms (which are tested by $u - v$)

$$\langle g(u) - g(v), u - v\rangle_X < \langle Au - Av, u - v\rangle_X + \langle g(u) - g(v), u - v\rangle_X = 0$$

If g is also strictly monotone, then the result follows. This is the case, e.g., for $g(u) = |u|^{p-2}u$ for $p > 1$.

3. (a) Let $\{\nabla p_n\}_{n\in\mathbb{N}}$ with $p_n \in L^2(\Omega)$ be a sequence, which converges in $H^{-1}(\Omega;\mathbb{R}^d)$. Since Ω is bounded we are able to assume $\int_\Omega p(x)dx = 0$ without changing ∇p_n. By the Nečas-Poincaré inequality, it holds for all $n, m \in \mathbb{N}$

$$\|p_n - p_m\|_{L^2(\Omega)} \leq C\|\nabla(p_n - p_m)\|_{H^{-1}(\Omega;\mathbb{R}^d)}.$$

Since $\{\nabla p_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $H^{-1}(\Omega;\mathbb{R}^d)$, we can deduce that $\{p_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega)$. Hence, there is a $p \in L^2(\Omega)$ such that

$$\|p_n - p\|_{L^2(\Omega)} \to 0 \text{ as } n \to \infty.$$

Therefore, by the definition of the gradient operator

$$\begin{aligned}
\|\nabla(p_n - p)\|_{H^{-1}(\Omega;\mathbb{R}^d)} &= \sup_{\|v\|_{H_0^1(\Omega;\mathbb{R}^d)}\leq 1} \left|\langle \nabla(p_n - p), v\rangle_{H_0^1(\Omega;\mathbb{R}^d)}\right| \\
&= \sup_{\|v\|_{H_0^1(\Omega;\mathbb{R}^d)}\leq 1} |(p_n - p, \mathrm{div} v)_{L^2(\Omega)}| \\
&\leq C\|p_n - p\|_{L^2(\Omega)} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Further, the norms $\|\nabla u\|_{H^{-1}(\Omega;\mathbb{R}^d)} + \|u\|_{H^{-1}(\Omega)}$ and $\|u\|_{L^2(\Omega)}$ are equivalent. Thus $\ker(\nabla)$ is finite and $\mathrm{im}(\nabla)$ is closed in $H^{-1}(\Omega;\mathbb{R}^d)$.

(b) The orthogonal spaces of Y and V are given by

$$\begin{aligned}
Y^\perp &= \{u \in H_0^1(\Omega;\mathbb{R}^d) : \overbrace{\langle \nabla p, u\rangle_{H_0^1(\Omega;\mathbb{R}^d)}}^{=-(p,\mathrm{div}u)_{L^2(\Omega)}} = 0 \text{ for all } p \in L^2(\Omega)\} \subset H_0^1(\Omega;\mathbb{R}^d), \\
V^\perp &= \{T \in H^{-1}(\Omega;\mathbb{R}^d) : \langle T, v\rangle_{H_0^1(\Omega;\mathbb{R}^d)} = 0 \text{ for all } v \in V\} \subset H^{-1}(\Omega;\mathbb{R}^d).
\end{aligned}$$

We have to prove if $F \in X^\perp$, then $F \in Y$. Since Y is closed in the reflexive space $H^{-1}(\Omega;\mathbb{R}^d)$ by (a), we can conclude $Y^{\perp\perp} = Y$. Thence, it is sufficient to prove $Y^\perp = X$ because we can conclude $X^\perp = Y^{\perp\perp} = Y$:

The direction $X \subset Y^\perp$ follows directly from their definitions. Now let $u \in Y^\perp \subset H_0^1(\Omega;\mathbb{R}^d)$. Then by definition of the weak gradient operator it holds

$$0 = \langle \nabla q, u\rangle_{H_0^1(\Omega;\mathbb{R}^d)} = -(q, \mathrm{div} u)_{L^2(\Omega)} \text{ for all } \; q \in L^2(\Omega).$$

Since div$u$ is an element in $L^2(\Omega)$, we choose $q := \mathrm{div} u$ as a test function yielding div$u = 0$ in $L^2(\Omega)$; hence $u \in X$.

(c) Suppose we have $q_1, q_2 \in L^2(\Omega)$ such that $F = \nabla q_1 = \nabla q_2$ in $H^{-1}(\Omega;\mathbb{R}^d)$. Then by the definition of the gradient operator

$$\int_\Omega (q_1(x) - q_2(x))\mathrm{div}\zeta(x)dx = 0 \text{ for all } \zeta \in H_0^1(\Omega;\mathbb{R}^d)$$

and for all $\zeta \in H_0^1(\Omega;\mathbb{R}^d)$

$$\begin{aligned} 0 = \sum_{i=1}^d \int_\Omega (q_1(x) - q_2(x))\partial_i\zeta_i(x)dx &= \sum_{i=1}^d \langle (q_1 - q_2), \partial_i\zeta_i\rangle_{H_0^1(\Omega)} \\ &= -\sum_{i=1}^d \langle \partial_i(q_1 - q_2), \zeta_i\rangle_{H_0^1(\Omega)}. \end{aligned}$$

By choosing successively test functions $\zeta \in C_c^\infty(\Omega;\mathbb{R}^d)$ in which two out of three components vanish, we have

$$\int_\Omega p_i(q_1 - q_2) = 0 \text{ in distributional sense for all } i \in \{1, 2, ..., d\}$$

and hence it holds $q_1 - q_2 =$ constant almost everywhere in Ω because Ω is connected.

(d) Let us apply deRham's theorem to the Stokes equations. For this we assume $f \in H^{-1}(\Omega;\mathbb{R}^d)$ and we look for a solution pair $(u,p) \in X \times L^2(\Omega)$ that solves

$$\begin{aligned} -\Delta u + \nabla p &= f \\ \mathrm{div} u &= 0 \\ \mathrm{tr} u &= 0 \end{aligned}$$

Applying divergence-free test functions $v \in X$ on the Stokes equations yields the variational problem

$$\int_\Omega \nabla u(x) \cdot \nabla v(x)dx = \langle f, v\rangle_{H_0^1(\Omega;\mathbb{R}^d)} \text{ for all } v \in X,$$

which has a unique solution $u \in X$ by Lax-Milgram's lemma. Hence

$$\langle \Delta u + f, v\rangle_{H_0^1(\Omega;\mathbb{R}^d)} = 0$$

for all $v \in X$ and by de Rham's theorem there is a $p \in L^2(\Omega)$ such that $\nu\Delta u + f = \nabla p$ in $H^{-1}(\Omega;\mathbb{R}^d)$.

(e) It is essential to assume that f is an element in $H^{-1}(\Omega;\mathbb{R}^d)$. It would be desirable to take a more general f, for example $f \in X'$ and it does not seem at first glance that it would lead to a problem. But X' is no distribution space and thus de Rham's theorem is not applicable. If a distributional pressure existed such that the Navier-Stokes equations were fulfilled in distributional sense, then we would have

$$f = -\Delta u + \nabla p \text{ in distributional sense.}$$

But this would contradict $X' \nsubseteq (C_c^\infty)'(\Omega;\mathbb{R}^d)$.