Adaptive FEM-BEM Coupling Method for Elasto-Plastic Analysis

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Abstract. In this paper we present an adaptive FEM-BEM coupling method for elasto-plastic analysis in which the nonlinearity, e.g., plastic material behavior, is treated by the FEM while large parts of the finite/infinite linear elastic body are treated using the BEM. A procedure that is easily automated is developed to generate and adapt the FEM zone of discretization (consequently the BEM sub-domain discretization), according to the state of computation. The adaptive FEM-BEM coupling method employs smaller FEM zones of discretization (FEM sub-domain).The adaptive coupling method eliminates the disadvantages of a prior definition and manual localization of the FEM and BEM sub-domains.

Introduction

The Finite Element Method (FEM) and the Boundary Element Method (BEM) are valuable and frequently used discretization techniques for obtaining approximate solutions to the partial differential equations that arise in scientific and engineering applications. The FEM, e.g., is especially well suited for the analysis of problems involving inhomogeneities or non-linear behavior, while the BEM has some advantages if stress singularities or unbounded sub-regions are present. It is conceptually and computationally very attractive to decompose the domain of the original problem and to use the most appropriate discretization method for the sub-domains under consideration. In this way, we are lead to the coupling of FEM and BEM (FEM-BEM coupling).

The first FEM-BEM coupling investigations date back to 1977 with the pioneering work of Zienkiewicz, Kelly and Bettes [1] based on a standard collocation BEM approach. Since then a large number of papers devoted to the topic have appeared. Due

to the unsymmetrical nature of the BEM technique used, the usefulness of the FEM-BEM coupling method has been limited. FEM-BEM coupling approaches based on the Symmetric Galerkin BEM (SGBEM) are quite recent; see, e.g., references [2-9].

The available coupling approaches necessitate a priori defined FEM and BEM zones of discretization (set by the user). Furthermore, the FEM and BEM zones remain unchanged during the computation. Unfortunately, a predefined FEM zone of discretization will probably result in either an under/overestimation of the nonlinear region where the FEM is employed. In the former case, inaccurate solutions is obtained to the problem at hand while for the later the computational cost may be higher than necessary.

This paper presents an adaptive FEM-BEM coupling method that is capabale of predicting zones sensible for FEM discretization. An outline of the paper is as follows. Section 2 briefly summarizes the basic SGBEM equations in elasticity, FEM equations in elasto-plasticity and the conventional (direct) and iterative FEM-BEM coupling methods. Then, in Section 3, we present the adaptive FEM-BEM coupling method. A numerical example that illustrates the advantages of the adaptive FEM-BEM coupling method is given in Section 4.

FEM-BEM Coupling

The system of boundary integral equations for a mixed boundary value problem in linear elasticity, may be written as follows

$$\begin{pmatrix} u \\ t \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K & V \\ D & \frac{1}{2}I + K' \end{pmatrix} \begin{pmatrix} u \\ t \end{pmatrix} \tag{1}$$

where V, K, and D are the single layer potential, double layer potential, hypersingular integral operators, respectively. In order to find the complete Cauchy data $[u, t]_\Gamma$, the first integral equation for $x \in \Gamma_D$ and the second one for $x \in \Gamma_N$ are rewritten as [10,11]

$$(Vt)(x) = \frac{1}{2} g(x) + (Ku)(x) \text{ for } x \in \Gamma_D \tag{2}$$

$$(Du)(x) = \frac{1}{2} h(x) - (K'u)(x) \text{ for } x \in \Gamma_N \tag{3}$$

where $g(x)$ and $h(x)$ are the given Dirichlet and Neumann data.

A Galerkin discretization is equivalent to the skew symmetric and positive definite system of linear equations

$$\begin{pmatrix} V_h & -K_h \\ K_h^T & D_h \end{pmatrix} \begin{pmatrix} \underline{t}^h \\ \underline{u}^h \end{pmatrix} = \begin{pmatrix} \underline{f}_1 \\ \underline{f}_2 \end{pmatrix} \tag{4}$$

where the block matrices in eq. (4) are given from discretization of the corresponding parts of the boundary.

In typical applications in linear elastostatics the Dirichlet part Γ_D, is often small compared to the Neumann part Γ_N where the boundary tractions are described. Therefore the inverse of the discrete single layer potential V_h may be computed using some direct method such as a Cholesky decomposition to obtain

$$\underline{t}^h = V_h^{-1} \left[\underline{f}_1 + K_h \underline{u}^h \right]. \tag{5}$$

Inserting eq (5) into the second of eq (4) yields the Schur complement system

$$\left[D_h + K_h^T V_h^{-1} K_h \right] \underline{u}^h = \underline{f}_2 - K_h^T V_h^{-1} \underline{f}_1. \tag{6}$$

The Schur complement system (6) is symmetric and positive definite and is suitable for coupling with FEM.

For a numerical representation of an arbitrary domain, Ω, with known boundary conditions specified at the entire boundary, $\Gamma = \Gamma_N \cup \Gamma_D$, the FEM and BEM are used. The domain is decomposed into two sub-domains, namely, ${}_F\Omega$ and ${}_B\Omega$ with the FEM-BEM coupling interface Γ_C. In all following equations subscripts F and B stand for *Finite Element* and *Boundary Element* sub-domains, respectively.
System (6) may be rewritten as

$$\begin{bmatrix} {}_B K_{CC} & {}_B K_{CB} \\ {}_B K_{BC} & {}_B K_{BB} \end{bmatrix} \begin{bmatrix} {}_B u_C \\ {}_B u \end{bmatrix} = \begin{bmatrix} {}_B f_C \\ {}_B f \end{bmatrix}. \tag{7}$$

For the FEM sub-domain, the assembled finite element equations in elasticity in partitioned form are

$$\begin{bmatrix} {}_F K_{FF} & {}_F K_{FC} \\ {}_F K_{CF} & {}_F K_{CC} \end{bmatrix} \begin{bmatrix} {}_F u \\ {}_F u_C \end{bmatrix} = \begin{bmatrix} {}_F f \\ {}_F f_C \end{bmatrix} \tag{8}$$

where ${}_F K$ is the complete FEM stiffness matrix.

The stiffness matrix ${}_B K$ can be interpreted as the element stiffness matrix of a finite macro element, computed by the BEM

$$\begin{bmatrix} {}_F K_{FF} & {}_F K_{FC} & \\ {}_F K_{CF} & {}_F K_{CC} + {}_B K_{CC} & {}_B K_{CB} \\ & {}_B K_{BC} & {}_B K_{BB} \end{bmatrix} \begin{bmatrix} {}_F u \\ u_C \\ {}_B u \end{bmatrix} = \begin{bmatrix} {}_F f \\ {}_B f_C + {}_F f_C \\ {}_B f \end{bmatrix}. \tag{9}$$

For an elasto-plastic analysis, the incremental form of the FEM equations, in a partitioned form are

$$\begin{bmatrix} \Delta\ {}_F \psi \\ \Delta_F\ \psi_C \end{bmatrix} = \begin{bmatrix} {}_F K_{T\,FF} & {}_F K_{T\,FC} \\ {}_F K_{T\ CF} & {}_F K_{T\,CC} \end{bmatrix} \begin{bmatrix} \Delta_F u \\ \Delta_F u_C \end{bmatrix} - \begin{bmatrix} \Delta_F f \\ \Delta_F f_C \end{bmatrix} \tag{10}$$

where K_T is the tangent stiffness matrix and $\Delta\psi$ is the residual (or out-of-balance) force vector. It should be noted that for each load increment, eq (10) are nonlinear and therefore are solved iteratively. Standard solution procedure, at each load increment, contains iterations over computations of tangent stiffness (based on current stress, and plastic strain, if required), applied loads based on current configuration, internal force and force residual. Then, displacement increment is calculated. With updated displacements the plastic strain increments at element integration points are obtained. Finally, check on convergence is carried out. If the procedure converged, plastic strains are updated and next increment proceeds.

Elasto-plastic problems with limited spread of plastic strains lend themselves to a coupled approach, where the plastic material behavior, is treated by the FEM while large parts of the finite/infinite linear elastic body are treated using the BEM. For each load increment, the following global equation systems are solved at each iteration

$$\begin{bmatrix} \Delta_F \psi \\ \Delta\psi_C \\ \Delta_B \psi \end{bmatrix} = \begin{bmatrix} {}_F K_{T\,FF} & {}_F K_{T\,FC} & \\ {}_F K_{T\ CF} & {}_F K_{T\,CC} + {}_B K_{CC} & {}_B K_{CB} \\ & {}_B K_{BC} & {}_B K_{BB} \end{bmatrix} \begin{bmatrix} \Delta_F u \\ \Delta u_C \\ \Delta_B u \end{bmatrix} - \begin{bmatrix} \Delta_F f \\ \Delta_B f_C + \Delta_F f_C \\ \Delta_B f \end{bmatrix}. \tag{11}$$

As an alternative to the conventional (direct) FEM-BEM coupling approach, a partitioned solution scheme can be used, where the systems of equations of the sub-domains are solved independently of each other. The interaction effects are taken into account as boundary conditions, which are imposed on the coupling interfaces. Iterations are performed in order to enforce satisfaction of the coupling conditions. Within the iteration procedure, a relaxation operator is applied to the interface boundary conditions in order to enable and speed up convergence. In this sense, the iterative coupling approaches are called interface relaxation FEM-BEM coupling methods. Interface relaxation FEM-BEM coupling methods in elasto-plasticity are discussed in details in reference [12].

Adaptive FEM-BEM Coupling Method

As mentioned earlier, it is not useful to predefine the FEM and BEM zones of discretization in an elasto-plastic FEM-BEM coupling analysis. The predefinition of the zones of discretization, will probably result in either an under/overestimation of the nonlinear region where the FEM is employed. More effective is a mechanism that allows an automated generation and adaption of the FEM zone of discretization. In order to avoid inaccurate or costly computations we propose in this section an adaptive FEM-BEM coupling method that automatically generate and progressively adapt the finite and boundary element zones of discretization. The adaptive FEM-BEM coupling method in elasto-plasticity follows the five basic steps:

1. Load increment and BEM elastic analysis with an initial BEM discretization.
   An hypothetical elastic stress state is determined
2. Detection of zones sensible for FEM discretization
   The hypothetical stress values computed at predefined points inside the BEM domain are checked against yielding (elastic prediction). Violation to the yield condition provides an initial estimate of the zones sensible for discretization by FEM (Fig. 1). For a final estimate of the FEM zones of discretization we propose to use simple fast post-calculations based on energetic methods, e.g., Neuber's and strain energy density methods [13]. This will account for relaxation and redistribution of stresses that occur due to plastic deformation.
3. Automatic generation of FEM zone of discretization (consequently the BEM sub domain discretization) for the current state of computation
   Particular regions that fulfil the proposed criterion are discretized by the FEM. In order to ensure the compatible coupling between the remaining BEM zone and the FEM zone, the interface is constructed reflecting the current situation. It is useful to reuse the BEM internal points as finite element nodes for the FEM discretization, as they are conveniently distributed in the particular area of interest. This will result in a reduction of the complexity of data management and ease of the automatic generation and adaption of the FEM zone of discretization.
4. Coupled FEM-BEM stress analysis involving elasto-plastic deformations is then conducted
5. Next load increment requires a repetition of steps 1-4.
   In our adaptive method, the user needs not to predefine the zones of discretization.

In the remainder of this section we will briefly elaborate on the post-calculations for the final estimate of the FEM zone of discretization. Let us consider materials of von-

Mises type obeying a multilinear strain hardening rule. Neuber's and strain energy density methods (Fig. 2) are energy equivalence between the hypothetical elastic and the elasto-plastic calculations of the same geometry submitted to the same loading [13]. For uni-dimensional states of stress, the product stress x strain in elasticity is assumed to be locally identical to the same product calculated by means of an elasto-plastic analysis. For tri-dimensional states of stress, the fundamental hypothesis may be written as

$$\sigma_{ij}\varepsilon_{ij} = \left(\sigma_{ij}\varepsilon_{ij}\right)_{elas} \tag{12}$$

where $\left(.\right)_{elas}$ corresponds to values determined from hypothetical elastic computations. The energy density balance, eq (12), is obtained by using the defined quantities appropriately for the actual elasto-plastic stress-strain state and the hypothetical elastic stress-strain state. However, a local method leads to a violation of equilibrium. Thus a proportionality factor is to be introduced in order to account for the stress relaxation and redistribution due to plastic deformations. From a virtual work principle we may utilize a global formulation

$$(\int_{\Omega} \sigma_{ij}\varepsilon_{ij}^{*} dV)_{\substack{elastic-plastic \\ computation}} \approx (\int_{\Omega} \sigma_{ij}\varepsilon_{ij}^{*} dV)_{\substack{elastic \\ computation}} \quad . \tag{13}$$

Based on the global formulation, eq (13), we propose a simple, yet effective, method for a final estimation of the FEM zones of discretization. The basic steps of post-calculations are summarized as

1. For regions that is initially predicted to yield (elastic prediction), compute the net values of the strain energy densities. This is achieved by subtracting the strain energy density that corresponds to the elastic limit from the hypothetical densities based on elastic BEM analysis
2. With the net values of strain energy densities of step 1, compute the net value of the strain energy (integrated over the initially estimated FEM discretization regions)
3. With the net strain energy that is vulnerable for redistribution computed in step 2 and the total hypothetical elastic strain energy of the whole domain, a pseudo value of the yielding strength is determined while assuming an equivalence of the actual tri-dimensional stress-strain state and a uni-dimensional state (Fig. 2)
4. For the whole domain, a final estimate of the FEM zone of discretization is achieved utilizing the strain energy yielding theory with a pseudo value of the material yield strength computed in step 3.

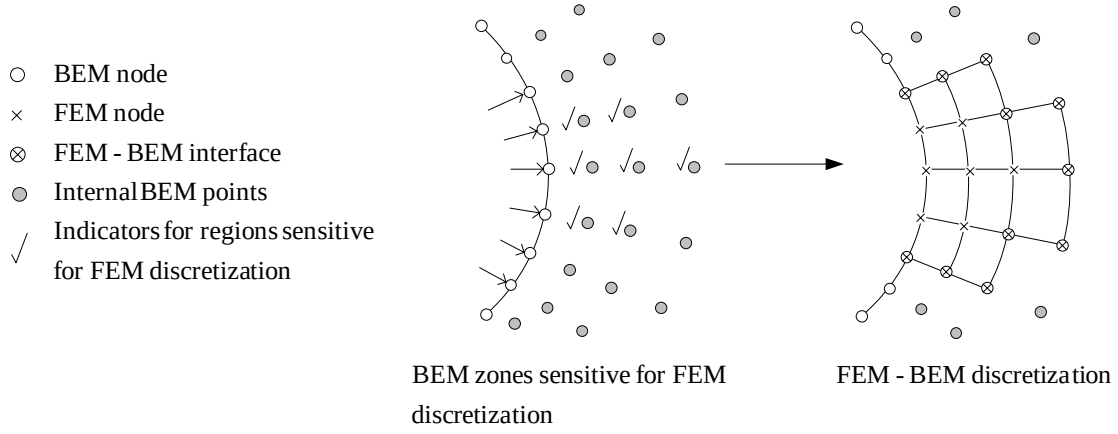

Fig. 1: Generation of FEM and BEM zones

The procedure outlined with its inherent assumptions, provide a simple, fast and effective method for a final estimation of the FEM and BEM zones of discretization. A usual FEM-BEM coupling analysis is then conducted while utilizing the finally estimated zones of discretization.

Numerical Example

In this section we present a numerical example, which serves as a benchmark problem in computational plasticity [14]. The benchmark problem is a stretched plate (width=height=200 *mm*) with a circular central hole (radius *r*=10 mm) under plane strain condition. A surface load P is applied on the plate's upper and lower edges. The applied tractions $\mathrm{P} = 100\ \mathrm{N/mm}^2$ are scaled with the load factor λ which is assumed to be as high as 4.5. Material properties of the plate are described by Young's modulus $E = 206.9\ \mathrm{GPa}$, Poisson's ratio $\nu = 0.29$. Material of Von Mises type is considered ($\sigma_y = 450\ \mathrm{MPa}$), with no hardening effect ($H = 0.$), as a yield function and plane strain loading conditions. Due to symmetry, only one quarter of the domain is discretized. The problem is solved by means of the adaptive algorithm presented in Section 3. The loads are applied incrementally. Fig. 3 shows the initial (elastic prediction) and final (energetic methods) estimates of the zones sensible for discretization by the FEM ($\lambda = 3.5$). Fig. 3 further shows the coupled FEM-BEM computed results (2000 quadrilateral finite elements and 357 boundary elements). The results compare well with those obtained by conventional FEM solutions (10,000 quadrilateral finite elements). The results clearly show the advantages of the adaptive coupled FEM-BEM models in terms of efficiency.

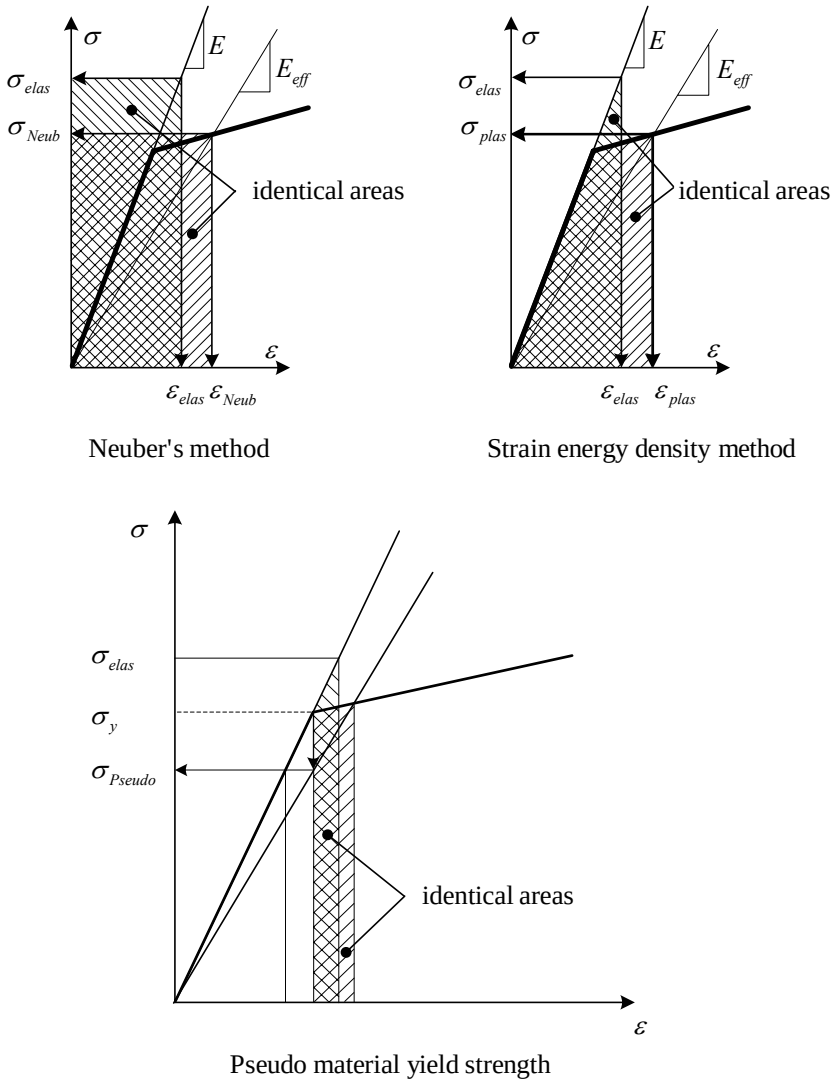

Fig. 2: Neuber's method, strain energy density method and a pseudo value of the yield strength based on hypothetical elastic computations

Conclusions

This paper deals with FEM-BEM coupling. The paper proposes the use of simple fast post-calculations, based on energetic methods and follows a simple hypothetical elastic boundary element computation, in order to give fast and helpful estimation of the FEM and BEM zones of discretization. The zones of discretization are progressively adapted according to the state of computation. The present adaptive coupling method eliminates the disadvantages of a prior definition and manual localization of the FEM and BEM sub-domains. It substantially decreases the size of FEM meshes, which plainly leads to reduction of required system resources and gain in efficiency.

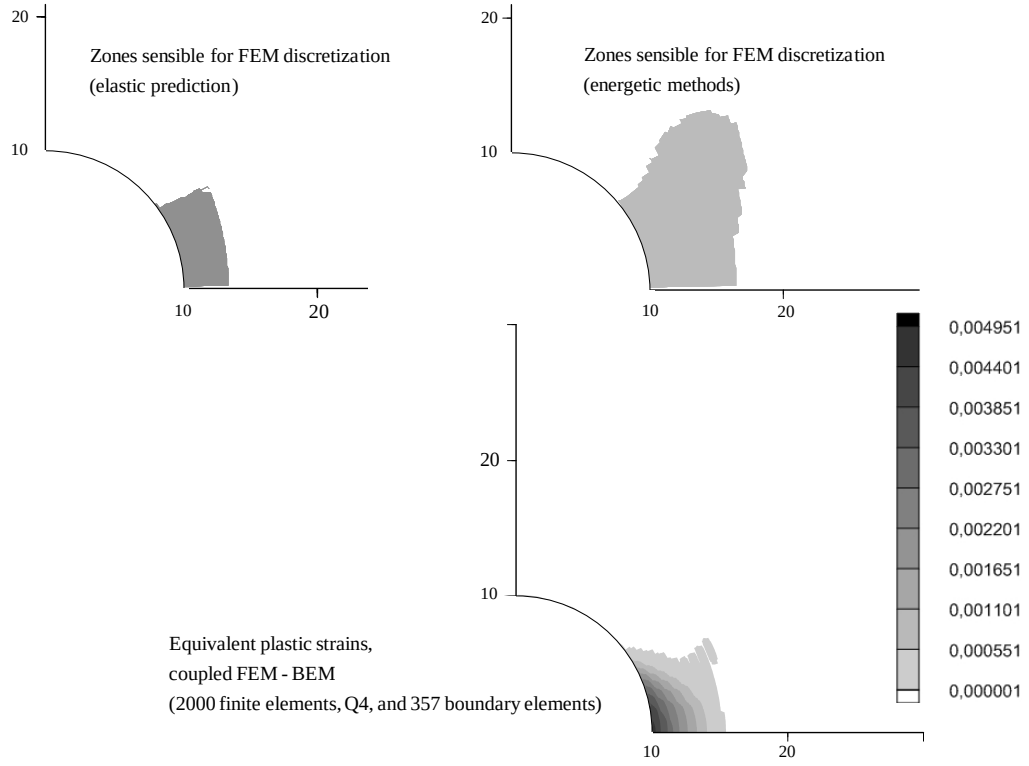

Fig. 3: Intial and final estimation of the FEM discretization zones and computed results via an adaptive FEM-BEM coupling method, $\lambda = 3.5$

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