Lecture notes
Monotone operators in nonlinear PDEs

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0 Introduction

In this chapter we want to generalize the following elementary result:

The function $F : \mathbb{R} \to \mathbb{R}$ fulfills the following conditions:
(a) F is monotonically increasing,
(b) F is continuous,
(c) F is coercive, i.e. $F(u) \to \pm\infty$ if $u \to \pm\infty$.
Then the equation $F(u) = b$ has a solution $u \in \mathbb{R}$ for all $b \in \mathbb{R}$. If F is strictly monotone, the solution u is uniquely determined.

This classical existence theorem follows from the intermediate value theorem for continuous functions. The theory of monotone operators, which applies this result to equations of the form

$$Au = b$$

in a reflexive Banach space X is based on some basic principles and tricks that we will briefly illustrate. Since it is easy to get lost in technical details, we will not go into them for now.

Theorem 0.1. *Let X be a separable, reflexive Banach space, and the operator $A : X \to X^*$*
(a) is monotone, i.e. for all $u, v \in X$ holds:

$$\langle Au - Av, u - v\rangle_X \geq 0$$

(b) hemicontinuous, i.e. the mapping

$$t \to \langle A(u + tv), w\rangle_X$$

is continuous in the interval $[0, 1]$, for all $u, v, w \in X$,
(c) coercive, i.e.

$$\lim_{\|u\|_X \to \infty} \frac{\langle Au, u\rangle_X}{\|u\|_X} = \infty$$

Then the main theorem about monotone operators states that A is surjective, i.e.

$$\forall b \in X^* \quad \exists u \in X: \quad Au = b.$$

Proof. The proof of this result essentially consists of the following steps:

(i) Galerkin approximation: Since X is separable, there is a basis $(w_i)_{i \in \mathbb{N}}$ of X, i.e. for $X_n := \operatorname{span}(w_1, \ldots, w_n)$ holds:

$$X = \overline{\bigcup_{n=1}^{\infty} X_n}$$

We approximate $Au = b$ by problems in the finite-dimensional spaces X_n to which Brouwer's fixed point theorem applies, which ensures the existence of a solution u_n for each of these problems.

(ii) Apriori estimation: We then show that the sequence of solutions (u_n) is bounded. This is based on the following argument: If $A : X \to X^*$ is coercive, then there exists an $R_0 > 0$, so that for all u with $\|u\|_X > R_0$ holds:

$$\langle Au, u \rangle_X \geq (1 + \|b\|_{X^*}) \|u\|_X$$

From this follows

$$\begin{aligned} \langle Au, u \rangle_X - \langle b, u \rangle_X &\geq (1 + \|b\|_{X^*}) \|u\|_X - \|b\|_{X^*} \|u\|_X \\ &\geq \|u\|_X > R_0 \end{aligned}$$

If $u \in X$, with $\|u\|_X > R_0$, were a solution of $Au = b$, then $0 \geq R_0 > 0$ would apply based on this calculation. However, this is a contradiction. Therefore, we obtain that every solution $u \in X$ of $Au = b$ corresponds to the a priori estimate

$$\|u\|_X \leq R_0$$

is sufficient.

(iii) Weak convergence: Since X is a reflexive Banach space, it follows from Eberlein-Šmuljan's theorem that a weakly convergent subsequence (u_n) can be selected from the bounded sequence (u_{n_k}), i.e.

$$u_{n_k} \rightharpoonup u \text{ in } X \quad (k \to \infty)$$

(iv) Existence of a solution: The limit u found in this way is a solution of the equation $Au = b$. We prove this statement using the Minty trick.

□

Lemma 0.2 (Minty's trick). *Let X be a Banach space and let $A : X \to X^*$ be a hemicontinuous, monotone operator. Then the following holds:*
(i) The operator A is maximally monotone, i.e. if for given $u \in X$, $b \in X^*$ the inequality

$$\langle b - Av, u - v \rangle_X \geq 0$$

holds for all $v \in X$, then $Au = b$.
(ii) A is of type M, i.e. from

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } X \quad (n \to \infty)\\ &Au_n \rightharpoonup b \quad \text{in } X^* \quad (n \to \infty)\\ &\limsup_{n\to\infty} \langle Au_n, u_n\rangle_X \le \langle b, u\rangle_X, \end{aligned}$$

follows $Au = b$.
(iii) From

$$u_n \rightharpoonup u \text{ in } X, \quad Au_n \to b \text{ in } X^* \quad (n \to \infty)$$

or alternatively

$$u_n \to u \text{ in } X, \quad Au_n \rightharpoonup b \quad \text{in } X^* \quad (n \to \infty)$$

follows $Au = b$.

Proof. ad (i): Let $u \in X$ and $b \in X^*$ be given so that the above assumption is satisfied. For any $w \in X$ we set $v := u - tw, t > 0$, and obtain the following implication based on the assumption:

$$\langle b - Av, u - v\rangle_X \ge 0 \quad \to \quad \langle b - A(u + t(-w)), w\rangle_X \ge 0$$

Since A is hemicontinuous, it follows from the limit transition $t \searrow 0^+$ that for all $w \in X$:

$$\langle b - Au, w\rangle_X \ge 0$$

We replace w with $-w$ and obtain the inverse inequality. Overall, $\langle b - Au, w\rangle_X = 0$ applies to all $w \in X$, i.e. $b = Au$.

ad (ii): Since A is monotone, it follows for all $v \in X, n \in \mathbb{N}$

$$0 \le \langle Au_n - Av, u_n - v\rangle_X = \langle Au_n, u_n\rangle_X - \langle Av, u_n\rangle_X - \langle Au_n - Av, v\rangle_X .$$

After applying the superior limit, we obtain the following for all $v \in X$ due to the preconditions

$$0 \le \langle b, u\rangle_X - \langle Av, u\rangle_X - \langle b - Av, v\rangle_X = \langle b - Av, u - v\rangle_X$$

Thus $Au = b$ follows from (i).

ad (iii): The assertion is a consequence of (ii) if we know that from

$$x_n \rightharpoonup x \text{ in } X, \quad f_n \to f \text{ in } X^* \quad (n \to \infty)$$

resp.

$$x_n \to x \text{ in } X, \quad f_n \rightharpoonup f \text{ in } X^* \quad (n \to \infty)$$

it follows that

$$\langle f_n, x_n\rangle_X \to \langle f, x\rangle_X \quad (n \to \infty).$$

In our situation, it follows that $\langle Au_n, u_n\rangle_X \to \langle b, u\rangle_X (n \to \infty)$. However, assertions (ii) and (iii) of the following lemma provide these statements. □

Lemma 0.3 (Principles of convergence)**.** *Let X be a Banach space. Then holds:*
(i) If $x_n \rightharpoonup x$ in X as $n \to \infty$, then there is a constant c such that $\|x_n\|_X \leq c$ for all $n \in \mathbb{N}$.
(ii) If

$$\begin{aligned} x_n &\rightharpoonup x \quad \text{in } X \quad (n \to \infty), \\ f_n &\to f \quad \text{in } X^* \quad (n \to \infty), \end{aligned}$$

then follows

$$\langle f_n, x_n \rangle_X \to \langle f, x \rangle_X \quad (n \to \infty)$$

(iii) If

$$\begin{aligned} x_n &\to x \quad \text{in } X \quad (n \to \infty), \\ f_n &\rightharpoonup f \quad \text{in } X^* \quad (n \to \infty), \end{aligned}$$

then follows

$$\langle f_n, x_n \rangle_X \to \langle f, x \rangle_X \quad (n \to \infty).$$

(iv) Let X be additionally reflexive. Let the sequence (x_n) be bounded. If all weakly convergent subsequences of (x_n) converge to the same limit x, then the entire sequence (x_n) converges weakly to x.

Proof. Exercises. □

1 Monotone operators

Definition 1.1. *Let X be a Banach space and $A : X \to X^*$ an operator. Then A is called*
(i) is monotone if and only if the following holds for all $u, v \in X$:

$$\langle Au - Av, u - v \rangle_X \geq 0.$$

(ii) strictly monotone if for all $u, v \in X, u \neq v$ holds:

$$\langle Au - Av, u - v \rangle_X > 0.$$

(iii) strongly monotone if and only if there is a $c > 0$ such that for all $u, v \in X$:

$$\langle Au - Av, u - v \rangle_X \geq c\|u - v\|_X^2$$

(iv) coercive exactly when

$$\lim_{\|u\|_X \to \infty} \frac{\langle Au, u \rangle_X}{\|u\|_X} = \infty$$

Remark 1.2. *(i) Obviously, the following implications apply:*
A is strongly monotone $\Rightarrow$ A is strictly monotone $\Rightarrow$ A is monotone.
(ii) If A is strongly monotone, then A is also coercive. In fact, the following holds:

$$\begin{aligned} \langle Au, u \rangle_X &= \langle Au - A(0), u \rangle_X + \langle A(0), u \rangle_X \\ &\geq c\|u\|_X^2 - \|A(0)\|_{X^*}\|u\|_X \end{aligned}$$

therefore follows

$$\frac{\langle Au, u\rangle_X}{\|u\|_X} \geq c\|u\|_X - \|A(0)\|_{X^*} \to \infty \quad \text{for } \|u\|_X \to \infty.$$

Example 1.3. *1. Given a function $f : \mathbb{R} \to \mathbb{R}$. We consider the function f as an operator from X to X^* with $X = \mathbb{R} = X^*$. In $\mathbb{R}$ the duality product is just the multiplication, i.e.*

$$\langle f(u) - f(v), u - v\rangle_X = (f(u) - f(v))(u - v).$$

Thus the following statements apply:
(i) $f : X \to X^*$ (strictly) monotone $\Leftrightarrow$ $f : \mathbb{R} \to \mathbb{R}$ (strictly) monotonically increasing.
(ii) f coercive $\Leftrightarrow \lim_{u\to\pm\infty} f(u) = \pm\infty$.
2. For the function $g : \mathbb{R} \to \mathbb{R}$

$$g(u) = \begin{cases} |u|^{p-2}u & \text{for } u \neq 0, \\ 0 & \text{for } u = 0, \end{cases}$$

it can be shown that holds:
(i) For $p > 1$ g is strictly monotone.
(ii) For $p \geq 2$ holds:

$$\langle g(u) - g(v), u - v\rangle_X \geq c|u - v|^p.$$

(iii) For $p = 2$ g is strongly monotone.

Definition 1.4. *Let X, Y be Banach spaces and let $A : X \to Y$ be an operator. Then A is called*
(i) is said to be completely continuous (weak-strong continuous) if and only if

$$u_n \rightharpoonup u \text{ in } X \quad (n \to \infty) \quad \Longrightarrow \quad Au_n \to Au \quad \text{in } Y \quad (n \to \infty)$$

(ii) demicontinuous (strong-weak continuous) when

$$u_n \to u \text{ in } X \quad (n \to \infty) \quad \Longrightarrow \quad Au_n \rightharpoonup Au \text{ in } Y \quad (n \to \infty)$$

(iii) hemicontinuous exactly if $Y = X^*$ and for all $u, v, w \in X$ the function

$$t \mapsto \langle A(u + tv), w\rangle_X$$

is continuous in the interval $[0, 1]$.
(iv) bounded if and only if A maps bounded sets in X into bounded sets in Y.
(v) locally bounded if and only if for all $u \in X$ there exists a $\varepsilon(u) > 0$ and a constant $K(u)$ such that for all $v \in X$ with $\|u - v\|_X \leq \varepsilon$ holds $\|Av\|_Y \leq K$.

Remark 1.5. *Obviously, the following implications hold:*
A is completely continuous $\Rightarrow$ A is continuous $\Rightarrow$ A is demicontinuous $\Rightarrow$ A is hemicontinuous.

$$A \text{ is bounded } \Rightarrow A \text{ is locally bounded.}$$

We now want to prove simple consequences of the above definitions.

Lemma 1.6. *Let X be a reflexive Banach space and $A : X \to X^*$ an operator. Then the following holds:*
(i) If A is completely continuous, then A is compact.
(ii) If A is demicontinuous, then A is locally bounded.
(iii) If A is monotone, then A is locally bounded.
(iv) If A is monotone and hemicontinuous, then A is demicontinuous.

Proof. ad (i): We want to show that for all bounded subsets $M \subseteq X$ the image set $A(M)$ is relatively sequence-compact. Let (Au_n) be any sequence from $A(M)$. Since M is bounded, (u_n) is also bounded. Due to the reflexivity of the space X, there is a weakly convergent subsequence (u_{n_k}), i.e. $u_{n_k} \rightharpoonup u$ in $X (k \to \infty)$. From this follows $Au_{n_k} \to Au$ in $X^*(k \to \infty)$, since A is completely continuous. Thus $A(M)$ is relatively compact in sequence, which is equivalent to the relative compactness of the set $A(M)$ in Banach spaces.

ad (ii): Proof by contradiction: Let A not be locally bounded, i.e. there is a $u \in X$ and a sequence $(u_n) \subseteq X$ with $u_n \to u$ in $X(n \to \infty)$ such that $\|Au_n\|_{X^*} \to \infty (n \to \infty)$. Since A is demicontinuous, it follows that $Au_n \rightharpoonup Au$ in $X^*(n \to \infty)$. Due to Lemma 0.3 (i), (Au_n) is bounded. But this is a contradiction. So A is locally bounded.

ad (iii): Proof by contradiction: If A is not locally bounded, then there is an $u \in X$ and a sequence $(u_n) \subseteq X$ with $u_n \to u$ in $X(n \to \infty)$ such that $\|Au_n\|_{X^*} \to \infty (n \to \infty)$. We set

$$a_n := \left(1 + \|Au_n\|_{X^*} \|u_n - u\|_X\right)^{-1}.$$

The monotonicity of A provides that for all $v \in X$:

$$\begin{aligned} 0 &\leq \langle Au_n - Av, u_n - v \rangle_X \\ &= \langle Au_n - Av, (u_n - u) + (u - v) \rangle_X \end{aligned}$$

With the above designation, this is equivalent to

$$\begin{aligned} a_n \langle Au_n, v - u \rangle_X &\leq a_n \left(\langle Au_n, u_n - u \rangle_X - \langle Av, u_n - v \rangle_X \right) \\ &\leq a_n \left(\|Au_n\|_{X^*} \|u_n - u\|_X + \|Av\|_{X^*} (\|u_n\|_X + \|v\|_X) \right) \\ &\leq 1 + c(v, u) \end{aligned}$$

where we use $a_n \leq 1$ and the boundedness of the sequence (u_n). If we replace v with $2u - v$ in this calculation, we also get

$$-a_n \langle Au_n, v - u \rangle_X \leq 1 + c(v, u)$$

Since $v \in X$ is arbitrary, $w := v - u$ is also an arbitrary point of X and we get for all $w \in X$

$$\sup_n |\langle a_n Au_n, w \rangle_X| \leq \tilde{c}(w, u) < \infty.$$

The continuous, linear mappings $a_n Au_n : X \to \mathbb{R}$ are pointwise bounded according to the above calculation. The principle of uniform boundedness thus yields

$$\sup_n \|a_n Au_n\|_{X^*} \leq c(u).$$

From this and from the definition of a_n we get

$$\|Au_n\|_{X^*} \leq \frac{c(u)}{a_n} = c(u)\left(1 + \|Au_n\|_{X^*}\|u_n - u\|_X\right).$$

Wegen $\|u_n - u\|_X \to 0(n \to \infty)$ there is a $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ $c(u)\|u_n - u\|_X < \frac{1}{2}$ holds and we obtain

$$\|Au_n\|_{X^*} \leq 2c(u).$$

Thus, the sequence $(\|Au_n\|_{X^*})$ is bounded, which is a contradiction to the assumption $\|Au_n\|_{X^*} \to \infty(n \to \infty)$ is. So the assertion is valid.

ad (iv): Let $(u_n) \subseteq X$ be a sequence with $u_n \to u$ in $X(n \to \infty)$. Since A is monotone, (iii) implies that A is locally bounded and thus (Au_n) is bounded. Due to the reflexivity of X, there is a subsequence (u_{n_k}) and an element $b \in X^*$ such that $Au_{n_k} \rightharpoonup b$ in $X^*(k \to \infty)$. By Lemma 0.2 (iii), we thus obtain $Au = b$, i.e. $Au_{n_k} \to Au$ in $X^*(k \to \infty)$. But all weakly convergent subsequences of (Au_n) converge weakly to Au, because otherwise there would be a subsequence with $Au_{n_l} \to c \neq b, (l \to \infty)$ in X^*. Lemma 0.2 (iii) again implies $Au = c$, which is a contradiction to $Au = b$. Thus, Lemma 0.3 (iv) provides that the entire sequence (Au_n) converges weakly to $b = Au$, i.e. A is demicontinuous. □

2 The theorem of Browder and Minty

We have now provided all the tools to prove the main theorem of the theory of monotone operators.

Theorem 2.1 (Browder–Minty). *Let X be a separable, reflexive Banach space. Furthermore, let $A : X \to X^*$ be a monotone, coercive, hemicontinuous operator. Then for all $b \in X^*$ there exists a solution $u \in X$ of*

$$Au = b.$$

The solution set is closed, bounded and convex. If A is strictly monotone, the solution is unique.

Proof. Due to the separability of X, there is a basis $(w_i)_{i\in\mathbb{N}}$ of X. We prove the theorem using the Galerkin method: To do this, we set

$$X_n := \operatorname{span}(w_1, \ldots, w_n)$$

and look for approximate solutions $u_n \in X_n$ of the form

$$u_n = \sum_{k=1}^{n} c_n^k w_k$$

which is the Galerkin system

$$\langle Au_n - b, w_k \rangle_X = 0, \quad k = 1, \ldots, n$$

solve.

(i) Solvability of discretized problem: We can solve elements $u_n \in X_n$ with vectors $\mathbf{c}_n := (c_n^1, \ldots, c_n^n)^\top \in \mathbb{R}^n$ can be identified. In particular, for $\mathbf{c} := (c^1, \ldots, c^n)^\top$

an equivalent norm is given by $|\mathbf{c}| := \left\|\sum_{k=1}^{n} c^k w_k\right\|_X$ on $\mathbb{R}^n$, which we will use in the following. Thus, the discretized problem can be viewed as a nonlinear system of equations for the vectors $\mathbf{c}_n \in \mathbb{R}^n$. We can do this using the mapping $\mathbf{g}_n := \left(g_n^1, \ldots, g_n^n\right)^\top : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$g_n^k : \mathbb{R}^n \to \mathbb{R} : \mathbf{c} \mapsto g_n^k(\mathbf{c}) := \left\langle A\left(\sum_{j=1}^{n} c^j w_j\right) - b, w_k\right\rangle_X, \quad k = 1, \ldots, n$$

rewrite into

$$\mathbf{g}_n\left(\mathbf{c}_n\right) = \mathbf{0}$$

According to Lemma 1.6 (iv), A is demicontinuous, since A is monotone and hemicontinuous. Therefore, the mapping $\mathbf{g}_n : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, since from $\mathbf{c}_l \to \mathbf{c}(l \to \infty)$ with respect to $|\cdot|$ in $\mathbb{R}^n$ it follows that $\sum_{j=1}^{n} c_l^j w_j$ converges to $\sum_{j=1}^{n} c^j w_j$ in X. It immediately follows that $\mathbf{g}_n\left(\mathbf{c}_l\right)$ converges against $\mathbf{g}_n(\mathbf{c})$ in the Euclidean norm and thus also with respect to the $|\cdot|$ norm.

Furthermore, for $\mathbf{c} = \left(c^1, \ldots, c^n\right)^\top$ and $v := \sum_{j=1}^{n} c^j w_j$

$$\sum_{k=1}^{n} g_n^k(\mathbf{c}) c^k = \langle Av, v\rangle_X - \langle b, v\rangle_X$$

Since A is coercive, i.e. $\frac{\langle Aw, w\rangle_X}{\|w\|_X} \to \infty\ (\|w\|_X \to \infty)$, there is an $R_0 > 0$ such that for all $\|w\|_X \geq R_0$ holds $\langle Aw, w\rangle_X \geq \|b\|_{X^*}\|w\|_X$. In particular, the following applies to c with $|\mathbf{c}| = \|v\|_X = R_0$

$$\langle Av, v\rangle_X \geq \|b\|_{X^*}\|v\|_X$$

and thus follows

$$\sum_{k=1}^{n} g_n^k(\mathbf{c}) c^k \geq \|b\|_{X^*}\|v\|_X - \|b\|_{X^*}\|v\|_X = 0$$

According to Brouwer's fixed point theorem, there is therefore a solution u_n of the Galerkin system with

$$\|u_n\|_X \leq R_0$$

In particular, the constant R_0 is independent of n, i.e. it is an a priori estimate.

(ii) Boundedness of (Au_n) : Since A is monotone, it follows from Lemma 1.6 (iii) that A is locally bounded. In particular, there are constants $r, M > 0$, so that the implication

$$\|w\|_X \leq r \quad \Rightarrow \quad \|Aw\|_{X^*} \leq M$$

is valid. Since u_n is a solution of the system, i.e. in particular $\langle Au_n, u_n\rangle_X = \langle b, u_n\rangle_X$, we obtain for all $n \in \mathbb{N}$

$$|\langle Au_n, u_n\rangle_X| \leq \|b\|_{X^*} \|u_n\|_X \leq \|b\|_{X^*} R_0.$$

Due to the monotonicity of A, for all $w \in X$:

$$\langle Au_n - Aw, u_n - w\rangle_X \geq 0$$

A scaled variant of the definition of the norm in X^* yields

$$\begin{aligned}
\|Au_n\|_{X^*} &= \sup_{\|w\|_X \leq r} \frac{1}{r} \langle Au_n, w\rangle_X \\
&\leq \sup_{\|w\|_X \leq r} \frac{1}{r} \left(\langle Aw, w\rangle_X + \langle Au_n, u_n\rangle_X - \langle Aw, u_n\rangle_X\right) \\
&\leq \frac{1}{r} \left(Mr + \|b\|_{X^*} R_0 + MR_0\right) < \infty
\end{aligned}$$

So the sequence $(Au_n) \subseteq X^*$ is bounded.

(iii) Convergence of the Galerkin method: Since X and X^* are reflexive and the sequences (u_n) and (Au_n) are bounded, there is a subsequence (u_{n_k}) with

$$\begin{aligned}
u_{n_k} &\rightharpoonup u \quad \text{in } X \\
Au_{n_k} &\rightharpoonup c \quad \text{in } X^* \quad (k \to \infty).
\end{aligned}$$

On the other hand, for all $w \in \bigcup_{l=1}^{\infty} X_l$ there is a $n_0 \in \mathbb{N}$ with $w \in X_{n_0}$. Since u_n is a solution of the Galerkin system, we obtain for all $n \geq n_0$

$$\langle Au_n, w\rangle_X = \langle b, w\rangle_X$$

from which follows

$$\lim_{n\to\infty} \langle Au_n, w\rangle_X = \langle b, w\rangle_X \quad \forall w \in \bigcup_{l=1}^{\infty} X_l$$

It follows that $\langle c - b, w\rangle_X = 0$ for all $w \in \bigcup_{l=1}^{\infty} X_l$. Since $\bigcup_{l=1}^{\infty} X_l$ is dense in X, it yields $b = c$ and thus

$$Au_{n_k} \rightharpoonup b \text{ in } X^* \quad (n \to \infty).$$

For the solution u_{n_k} of the Galerkin system, in particular $\langle Au_{n_k}, u_{n_k}\rangle_X = \langle b, u_{n_k}\rangle_X$, from which follows:

$$\lim_{k\to\infty} \langle Au_{n_k}, u_{n_k}\rangle_X = \lim_{k\to\infty} \langle b, u_{n_k}\rangle_X = \langle b, u\rangle_X.$$

The conditions of the Minty trick, Lemma 0.2 (ii), are therefore fulfilled and we obtain $Au = b$, i.e. u is a solution of the original operator equation $Au = b$.

(iv) Properties of the solution set: For a given $b \in X^*$ we set $S := \{u \in X \mid Au = b\}$. Then S has the following properties:

(a) $S \neq \emptyset$: This has just been proved.

(b) S is convex: Let $u_1, u_2 \in S$, i.e. $Au_i = b$ for $i = 1, 2$. For the convex combination $w = tu_1 + (1-t)u_2, t \in [0,1]$, and any $v \in X$ holds:

$$\begin{aligned}\langle b - Av, w - v\rangle_X &= \langle b - Av, tu_1 + (1-t)u_2 - (t + 1 - t)v\rangle_X \\ &= \langle b - Av, t\,(u_1 - v)\rangle_X + \langle b - Av, (1-t)\,(u_2 - v)\rangle_X \\ &= t\,\langle Au_1 - Av, u_1 - v\rangle_X + (1-t)\,\langle Au_2 - Av, u_2 - v\rangle_X \\ &\geq 0\end{aligned}$$

due to the monotonicity of A. An application of the Minty trick (cf. Lemma 0.2 (i)) yields $Aw = b$, i.e. $w \in S$. Thus S is convex.

(c) S is bounded: This follows from the coercivity of A. If S were not bounded, there would be an $u \in S$ for all $R > 0$ with $\|u\|_X \geq R > 0$. But analogous to the argumentation in step 1, we have

$$0 = \langle Au, u\rangle_X - \langle b, u\rangle_X \geq \|u\|_X > 0$$

However, this is a contradiction and therefore there is an $R_0 > 0$, so that for all $u \in S$ $\|u\|_X \leq R_0$.

(d) S is closed: For a sequence $(u_n) \subseteq S$, i.e. $Au_n = b$, with $u_n \to u$ in $X(n \to \infty)$, and for all $v \in X$ we have:

$$\begin{aligned}\langle b - Av, u - v\rangle_X &= \lim_{n\to\infty} \langle b - Av, u_n - v\rangle_X \\ &= \lim_{n\to\infty} \langle Au_n - Av, u_n - v\rangle_X \geq 0,\end{aligned}$$

due to the monotonicity of A. Using the Minty trick (cf. Lemma 0.2 (i)), $Au = b$ follows, i.e. $u \in S$.

(v) Uniqueness: Let A be strictly monotone. If there are two solutions $u \neq v$ of the operator equation, then on the one hand we have $Au = b = Av$ and on the other hand from the strict monotonicity of A follows

$$0 < \langle Au - Av, u - v\rangle_X = \langle b - b, u - v\rangle_X = 0.$$

This is a contradiction. So the equation can have at most one solution.

□

Corollary 2.1.1. *Let X be a separable, reflexive Banach space and let $A : X \to X^*$ be a strictly monotone, coercive, hemicontinuous operator. Then there exists the operator $A^{-1} : X^* \to X$ exists and is strictly monotone and demicontinuous.*

Proof. Exercises. □

3 The Nemyckii operator

In order to apply Browder and Minty's Theorem 2.1 to differential equations, we need the so-called Nemyckii operator

$$(F\mathbf{u})(x) := f(x, \mathbf{u}(x))$$

where $\mathbf{u} = (u^1, \ldots, u^n)^\top, \mathbf{u} : G \subseteq \mathbb{R}^N \to \mathbb{R}^n$, with a domain $G \subseteq \mathbb{R}^N$.
With respect to the function $f : G \times \mathbb{R}^n \to \mathbb{R}$ we make the following assumptions:
(i) Carathéodory condition:

$$\begin{aligned} f(\cdot, \boldsymbol{\eta}) : x \mapsto f(x, \boldsymbol{\eta}) \quad &\text{is measurable on } G \text{ for all } \boldsymbol{\eta} \in \mathbb{R}^n, \\ f(x, \cdot) : \boldsymbol{\eta} \mapsto f(x, \boldsymbol{\eta}) \quad &\text{is continuous on } \mathbb{R}^n \text{ for almost all } x \in G. \end{aligned}$$

(ii) Growth condition:

$$|f(x, \boldsymbol{\eta})| \leq |a(x)| + b \sum_{i=1}^{n} \left|\eta^i\right|^{p_i/q}$$

where $b > 0$ is a constant, $a \in L^q(G), 1 \leq q < \infty$ and $p_i \in [1, \infty)$, $i = 1, \ldots, n$.

Lemma 3.1. *Under the above assumptions on the function f and the set G, the Nemyckii operator*

$$F : \prod_{i=1}^{n} L^{p_i}(G) \to L^q(G)$$

is continuous and bounded. The following applies for all $\mathbf{u} \in \prod_{i=1}^{n} L^{p_i}(G)$ the estimate applies:

$$\|F\mathbf{u}\|_q \leq c \left(\|a\|_q + \sum_{i=1}^{n} \left\|u^i\right\|_{p_i}^{p_i/q} \right)$$

Proof. We only consider the case $n = 1, u = u_1, p = p_1$. The general case follows analogously.

(i) Measurability of Fu : Since $u \in L^p(G)$, the function $x \mapsto u(x)$ is Lebesgue-measurable on G. So there is a sequence (u_n) of staircase functions with

$$u_n \to u \text{ almost everywhere in } G \quad (n \to \infty)$$

Therefore, for almost all $x \in G$:

$$(Fu)(x) = f(x, u(x)) = \lim_{n\to\infty} f\left(x, u_n(x)\right)$$

since f is continuous in the second variable due to the Carathéodory condition (i). Since (u_n) are staircase functions, we have

$$f\left(x, u_n(x)\right) = f\left(x, \sum_{j=0}^{M(n)} c_j^n \chi_{G_j^n}(x)\right) = \sum_{j=0}^{M(n)} f\left(x, c_j^n\right) \chi_{G_j^n}(x)$$

with $c_0^n = 0$ and $G_0^n = G \backslash \bigcup_{i=1}^{M(n)} G_i^n$. Thus, $f(x, u_n(x))$ is measurable, since both the functions $f\left(x, c_j^n\right)$ and the characteristic functions $\chi_{G_j^n}$ are measurable. Furthermore, the limit value of measurable functions is measurable and therefore also Fu.

(ii) Boundedness of F : It applies to all $u \in L^p(G)$:

$$\begin{aligned}\|Fu\|_q^q &= \int_G |f(x,u(x))|^q dx \leq \int_G \left(|a(x)| + b|u(x)|^{p/q}\right)^q dx \\ &\leq C \int_G |a(x)|^q + b^q |u(x)|^p dx \leq C\left(\|a\|_q^q + \|u\|_p^p\right)\end{aligned}$$

where the growth condition (ii) and the following equivalence in $\mathbb{R}^M$

$$c\left(\sum_{i=1}^M |\xi_i|^r\right)^{\frac{1}{r}} \leq \sum_{i=1}^M |\xi_i| \leq C\left(\sum_{i=1}^M |\xi_i|^r\right)^{\frac{1}{r}}$$

which is valid for $r > 0$, were used. So F is limited and fulfills the estimate from before.

(iii) Continuity of $F : L^p(G) \to L^q(G)$: Let (u_n) be a sequence with $u_n \to u$ in $L^p(G)(n \to \infty)$. If F were not continuous, there would be a $\varepsilon > 0$ and a subsequence (u_{n_k}), so that for all $k \in \mathbb{N}$ applies

$$\|Fu_{n_k} - Fu\|_q \geq \varepsilon.$$

From $u_{n_k} \to u$ in $L^p(G)(k \to \infty)$ follows the existence of another subsequence $\left(u_{n_{k_\ell}}\right)$ with $u_{n_{k_\ell}} \to u$ almost everywhere in $G(\ell \to \infty)$ and it holds:

$$\begin{aligned}\left|f\left(x, u_{n_{k_\ell}}(x)\right) - f(x,u(x))\right|^q &\leq C\left(\left|f\left(x, u_{n_{k_\ell}}(x)\right)\right|^q + |f(x,u(x))|^q\right) \\ &\leq C\left(|a(x)|^q + b^q \left|u_{n_{k_\ell}}(x)\right|^p + |f(x,u(x))|^q\right) \\ &=: h_{n_{k_\ell}}(x)\end{aligned}$$

where the growth condition (ii) was used. For the integrands $\left(h_{n_{k_\ell}}\right) \subseteq L^1(G)$ on the right-hand side, $h(x) := C\left(|a(x)|^q + b^q|u(x)|^p + |f(x,u(x))|^q\right)$:

$$\begin{aligned} h_{n_{k_\ell}}(x) &\to h(x) && \text{almost everywhere in } G \quad (\ell \to \infty), \\ \int_G h_{n_{k_\ell}}(x)dx &\to \int_G h(x)dx && (\ell \to \infty),\end{aligned}$$

since $u_n \to u$ in $L^p(G)(n \to \infty)$, thus $\|u_n\|_p \to \|u\|_p(n \to \infty)$. Furthermore, $\left|f\left(x, u_{n_{k_\ell}}(x)\right) - f(x,u(x))\right|^q \to 0$ for almost all $x \in G(\ell \to \infty)$, since f is continuous in the second variable due to the Carathéodory condition (i). Thus, the generalized theorem of dominated convergence together with the definition of F yields

$$\left\|Fu_{n_{k_\ell}} - Fu\right\|_q^q \to 0 \quad (\ell \to \infty)$$

This is a contradiction and therefore $F : L^p(G) \to L^q(G)$ is continuous.

□

4 Quasilinear elliptic equations

As an application of Browder and Minty's Theorem 2.1 and the Nemyckii operator, we consider the following boundary value problem for a quasilinear elliptic equation:

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + su &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Let $1 < p < \infty, \Omega$ be a bounded domain in $\mathbb{R}^d$ with $\partial\Omega \in C^{0,1}$ and $s \geq 0$. If we formally multiply the PDE by u, integrate over Ω and partially integrate, we obtain the following a priori estimate

$$\int_\Omega |\nabla u|^p + s|u|^2 dx \leq c(f)$$

From this we see that for all $s \geq 0$ the canonical Sobolev space for the analysis is $W_0^{1,p}(\Omega)$. Note, however, that for $s > 0$ the space $W_0^{1,p}(\Omega) \cap L^2(\Omega)$ is also a natural choice.
The weak formulation of problem is: For a given $f \in L^p(\Omega)^*$ we look for $u \in X := W_0^{1,p}(\Omega)$ such that for all $\varphi \in X$

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi + su\varphi dx = \int_\Omega f\varphi dx$$

Therefore, we define an operator A by

$$\langle Au, \varphi\rangle_X := \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi + su\varphi dx$$

and a functional b by

$$\langle b, \varphi\rangle_X := \int_\Omega f\varphi dx, \quad \forall\varphi \in X$$

Remark 4.1. *For $p \in (1, 2)$ one have to be careful with the integrals, because at points $x \in \Omega$ with $\nabla u(x) = \mathbf{0}$ the expression $|\nabla u(x)|^{p-2}$ is not defined. Therefore, $|\nabla u|^{p-2}\nabla u$ as a unit, since for all $p \in (1, \infty)$ the function*

$$\mathbf{g} = \left(g^1, \ldots, g^d\right)^\top : \mathbb{R}^d \backslash \{\mathbf{0}\} \to \mathbb{R}^d : \boldsymbol{\zeta} \mapsto |\boldsymbol{\zeta}|^{p-2}\boldsymbol{\zeta}$$

can be continued continuously on the entire $\mathbb{R}^d$ by $\mathbf{g}(\mathbf{0}) := \mathbf{0}$. We use this convention as well as in the further presentation.

Lemma 4.2. *Let Ω be a bounded domain of $\mathbb{R}^d$ with Lipschitz continuous boundary $\partial\Omega$. Furthermore, let $f \in L^{p'}(\Omega), p' = \frac{p}{p-1}, p \in (1, \infty)$ and $s \geq 0$. For $p \geq \frac{2d}{d+2}$ the operator A maps the space $X = W_0^{1,p}(\Omega)$ into its dual space, i.e. $A \colon X \to X^*$, and is bounded. The* *functional b is an element of X^*. Furthermore, the weak formulation is equivalent to the operator equation in X^**

$$Au = b.$$

Proof. We set $X := W_0^{1,p}(\Omega)$ and $\|u\|_X := \|\nabla u\|_p$. Due to the "zero boundary conditions", this norm is equivalent to the usual $W_0^{1,p}(\Omega)$ norm $\|u\|_{W_0^{1,p}} = \left(\int_\Omega |u|^p + |\nabla u|^p dx\right)^{\frac{1}{p}}$.

(i) $A : X \to X^*$: It applies to $u, \varphi \in X$:

$$\begin{aligned}
|\langle Au, \varphi\rangle_X| &\leq \int_\Omega |\nabla u|^{p-1}|\nabla\varphi| dx + s\int_\Omega |u\varphi| dx \\
&\leq \left(\int_\Omega |\nabla u|^{(p-1)p'} dx\right)^{\frac{1}{p'}} \left(\int_\Omega |\nabla\varphi|^p dx\right)^{\frac{1}{p}} \\
&\quad + s\left(\int_\Omega |u|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |\varphi|^2 dx\right)^{\frac{1}{2}} \\
&= \|\nabla u\|_p^{p-1}\|\nabla\varphi\|_p + s\|u\|_2\|\varphi\|_2,
\end{aligned}$$

where we have used the Hölder inequality and $p' = \frac{p}{p-1}$. For $1 \leq p < d$ the Sobolev embedding $X = W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ applies with $q \leq \frac{dp}{d-p}$. In particular, $X \hookrightarrow L^2(\Omega)$ applies if $2 \leq \frac{dp}{d-p} \leftrightarrow p \geq \frac{2d}{d+2}$. If $p \geq d$, we use the Sobolev embeddings $X \hookrightarrow W^{1,d}(\Omega) \hookrightarrow L^q(\Omega)$, which are valid for all $q < \infty$. So we get that for $p \geq \frac{2d}{d+2}$ and all $\varphi \in X$ holds:

$$\|\varphi\|_2 \leq c_1\|\varphi\|_X = c_1\|\nabla\varphi\|_p$$

Overall, this results in

$$\begin{aligned}
|\langle Au, \varphi\rangle_X| &\leq \|\nabla u\|_p^{p-1}\|\nabla\varphi\|_p + s\|u\|_2\|\varphi\|_2 \\
&\leq c\left(\|\nabla u\|_p^{p-1} + s\|\nabla u\|_p\right)\|\nabla\varphi\|_p
\end{aligned}$$

Based on the definition of the norm of Au in X^* we have

$$\|Au\|_{X^*} = \sup_{\|\varphi\|_X \leq 1} |\langle Au, \varphi\rangle_X| \leq c\left(\|\nabla u\|_p^{p-1} + s\|\nabla u\|_p\right),$$

and thus $Au \in X^*$ and $A : X \to X^*$, provided that $p \geq \frac{2d}{d+2}$. We immediately see that the operator A is bounded.

(ii) Using the Hölder inequality and the definition of the dual norm, we obtain

$$\begin{aligned}
\|b\|_{X^*} &= \sup_{\|\varphi\|_X \leq 1} |\langle b, \varphi\rangle_X| \leq \sup_{\|\varphi\|_X \leq 1} \|f\|_{p'}\|\varphi\|_p \\
&\leq c\|f\|_{p'}
\end{aligned}$$

for $p \geq 1$, since $X = W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$, i.e. $\|\varphi\|_p \leq c\|\varphi\|_X$.

(iii) From steps 1 and 2, as well as the definitions of A and b, it follows that the weak formulation is just

$$\langle Au, \varphi\rangle_X = \langle b, \varphi\rangle_X \quad \forall \varphi \in X$$

is. But this is the operator equation $Au = b$ in X^*.

□

Remark 4.3. *For $s = 0$ the restriction $p \geq \frac{2d}{d+2}$ is not necessary in the previous lemma. If one works for $s > 0$ with $X = W_0^{1,p}(\Omega) \cap L^2(\Omega)$ provided with the norm $\|u\|_X := \|\nabla u\|_p + \|u\|_2$, the restriction $p \geq \frac{2d}{d+2}$ is also omitted.*

Lemma 4.4. *Under the conditions of Lemma 4.2, the operator $A : X \to X^*$ is strictly monotone, coercive and continuous.*

Proof. (i) A is strictly monotone: The operator A can be understood using the function $\mathbf{g}$ defined as before. For $i, j = 1, \ldots, d$ and $\boldsymbol{\zeta} \neq \mathbf{0}$ we have

$$\partial_j g^i(\boldsymbol{\zeta}) = |\boldsymbol{\zeta}|^{p-2}\delta_{ij} + (p-2)|\boldsymbol{\zeta}|^{p-4}\zeta^i\zeta^j$$

and thus for all $\boldsymbol{\zeta} \in \mathbb{R}^d\backslash\{\mathbf{0}\}, \boldsymbol{\eta} \in \mathbb{R}^d, 1 < p < \infty$,

$$\begin{aligned}\sum_{i,j=1}^d \partial_j g^i(\boldsymbol{\zeta})\eta^i\eta^j &= |\boldsymbol{\zeta}|^{p-2}\left(|\boldsymbol{\eta}|^2 + (p-2)\frac{(\boldsymbol{\zeta}\cdot\boldsymbol{\eta})^2}{|\boldsymbol{\zeta}|^2}\right)\\ &\geq \min(1, p-1)|\boldsymbol{\zeta}|^{p-2}|\boldsymbol{\eta}|^2\end{aligned}$$

To prove the strict monotonicity of A we want to use the main theorem of differential and integral calculus. Since g only belongs to the space $C^0\left(\mathbb{R}^d\right) \cap C^1\left(\mathbb{R}^d\backslash\{\mathbf{0}\}\right)$, we approximate $\mathbf{g}$ by $\mathbf{g}_\varepsilon(\boldsymbol{\zeta}) := (\varepsilon^2 + |\boldsymbol{\zeta}|^2)^{\frac{p-2}{2}}\boldsymbol{\zeta}$, $\boldsymbol{\zeta} \in \mathbb{R}^d, \varepsilon > 0$. Obviously, $\mathbf{g}_\varepsilon \in C^1\left(\mathbb{R}^d\right)$, as well as $\mathbf{g}_\varepsilon(\boldsymbol{\zeta}) \to \mathbf{g}(\boldsymbol{\zeta})$ for all $\boldsymbol{\zeta} \in \mathbb{R}^d$, $\nabla\mathbf{g}_\varepsilon(\boldsymbol{\zeta}) \to \nabla\mathbf{g}(\boldsymbol{\zeta})$ for all $\boldsymbol{\zeta} \in \mathbb{R}^d\backslash\{\mathbf{0}\}$ and $|\nabla\mathbf{g}_\varepsilon(\boldsymbol{\zeta})| \leq c(p,d)|\boldsymbol{\zeta}|^{p-2}$ for all $\zeta \in \mathbb{R}^d$. It can be shown (exercise) that there are constants that depend only on p, so that for all $|\boldsymbol{\zeta}| + |\boldsymbol{\eta}| > 0, p \in (1, \infty)$, the following applies

$$c(|\boldsymbol{\zeta}| + |\boldsymbol{\eta}|)^{p-2} \leq \int_0^1 |\boldsymbol{\zeta} + \tau(\boldsymbol{\eta} - \boldsymbol{\zeta})|^{p-2}d\tau \leq C(|\boldsymbol{\zeta}| + |\boldsymbol{\eta}|)^{p-2}$$

The main theorem now returns for all $\boldsymbol{\zeta} \neq \boldsymbol{\eta}$

$$\begin{aligned}\mathbf{g}_\varepsilon(\boldsymbol{\zeta}) - \mathbf{g}_\varepsilon(\boldsymbol{\eta}) &= \int_0^1 \frac{d}{d\tau}\mathbf{g}_\varepsilon(\boldsymbol{\eta} + \tau(\boldsymbol{\zeta} - \boldsymbol{\eta}))d\tau\\ &= \int_0^1 \nabla\mathbf{g}_\varepsilon(\boldsymbol{\eta} + \tau(\boldsymbol{\zeta} - \boldsymbol{\eta}))\cdot(\boldsymbol{\zeta} - \boldsymbol{\eta})d\tau\end{aligned}$$

The above properties of $\mathbf{g}$ and $\mathbf{g}_\varepsilon$, as well as the theorem on dominated convergence provide for $\varepsilon \to 0$

$$\mathbf{g}(\boldsymbol{\zeta})-\mathbf{g}(\boldsymbol{\eta})=\int_0^1 \nabla \mathbf{g}(\boldsymbol{\eta}+\tau(\boldsymbol{\zeta}-\boldsymbol{\eta}))\cdot(\boldsymbol{\zeta}-\boldsymbol{\eta}) d\tau$$

For any $u \neq v \in X$ we get with $s \geq 0$

$$\begin{aligned}
&\langle Au-Av,u-v\rangle_X\\
&=\int_\Omega \sum_{i=1}^d \left(g^i(\nabla u)-g^i(\nabla v)\right)(\partial_i u-\partial_i v)\, dx+s\int_\Omega |u-v|^2 dx\\
&\geq \int_\Omega \int_0^1 \sum_{i,j=1}^d \partial_j g^i(\nabla v+\tau(\nabla u-\nabla v))\,(\partial_j u-\partial_j v)\,(\partial_i u-\partial_i v)\, d\tau dx\\
&\geq c\int_\Omega |\nabla u-\nabla v|^2\int_0^1 |\nabla v+\tau(\nabla u-\nabla v)|^{p-2} d\tau dx\\
&\geq c\int_\Omega |\nabla u-\nabla v|^2(|\nabla u|+|\nabla v|)^{p-2}dx>0,
\end{aligned}$$

i.e. A is strictly monotone.

(ii) A is coercive: We have for $u \in X$

$$\langle Au,u\rangle_X=\int_\Omega |\nabla u|^p+s|u|^2 dx=\|\nabla u\|_p^p+s\|u\|_2^2\geq \|\nabla u\|_p^p$$

therefore follows

$$\frac{\langle Au,u\rangle_X}{\|u\|_X}\geq \|\nabla u\|_p^{p-1}\to\infty \quad (\|u\|_X\to\infty)$$

if $p>1$.

(iii) A is continuous: Let $(u_n)\subseteq X$ be a sequence with $u_n \to u$ in $X (n\to\infty)$, i.e. in particular $\nabla u_n \to \nabla u$ in $L^p\left(\Omega;\mathbb{R}^d\right)(n\to\infty)$. We set

$$\mathbf{F}(\nabla u)(x):=\mathbf{g}(\nabla u(x)).$$

Since $\mathbf{g}$ componentizes the estimate

$$\left|g^i(\boldsymbol{\zeta})\right|\leq |\boldsymbol{\zeta}|^{p-1}=|\boldsymbol{\zeta}|^{\frac{p}{q}},\quad i=1,\ldots,d$$

is satisfied with $q=\frac{p}{p-1}$, $\mathbf{F}$ is a vector-valued Nemyckii operator. It therefore follows from Lemma 3.1 that $\mathbf{F}: L^p\left(\Omega;\mathbb{R}^d\right)\to L^{p'}\left(\Omega;\mathbb{R}^d\right)$ is continuous, i.e. for our sequence (u_n) holds:

$$\mathbf{F}(\nabla u_n)\to\mathbf{F}(\nabla u) \text{ in } L^{p'}\left(\Omega;\mathbb{R}^d\right)\quad (n\to\infty)$$

Thus we get

$$\begin{aligned}\langle Au_n - Au, \varphi\rangle_X &= \int_\Omega (\mathbf{F}(\nabla u_n) - \mathbf{F}(\nabla u)) \cdot \nabla\varphi dx + s\int_\Omega (u_n - u)\varphi dx \\ &\leq \|\mathbf{F}(\nabla u_n) - \mathbf{F}(\nabla u)\|_{p'} \|\nabla\varphi\|_p + s\|u_n - u\|_2 \|\varphi\|_2 \\ &\leq c\left(\|\mathbf{F}(\nabla u_n) - \mathbf{F}(\nabla u)\|_{p'} + \|u_n - u\|_X\right)\|\varphi\|_X,\end{aligned}$$

since $X = W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$ for $p \geq \frac{2d}{d+2}$. Based on the definition of the norm in dual space, it follows that

$$\|Au_n - Au\|_{X^*} \leq c\left(\|\mathbf{F}(\nabla u_n) - \mathbf{F}(\nabla u)\|_{p'} + \|u_n - u\|_X\right).$$

For $n \to \infty$ the right-hand side converges to 0, since $u_n \to u$ in $X (n \to \infty)$ and $\mathbf{F}(\nabla u_n) \to \mathbf{F}(\nabla u)$ in $L^{p'}(\Omega; \mathbb{R}^d)$ $(n \to \infty)$, i.e. the operator A is continuous.

□

Theorem 4.5. *Let Ω be a bounded domain of $\mathbb{R}^d$ with Lipschitz continuous boundary $\partial\Omega$ and let $s \geq 0$. For $p \geq \frac{2d}{d+2}, p \in (1, \infty)$ and all $f \in L^{p'}(\Omega)$, $p' = \frac{p}{p-1}$, there exists exactly one weak solution u of the boundary value problem.*

Proof. The space $X = W_0^{1,p}(\Omega)$ is a separable and reflexive Banach space. Thus it follows from Lemmas 4.2 and 4.4 that we can apply Browder–Minty's Theorem 2.1, which immediately yields the assertion. □

Remark 4.6. *(i) The restriction $p \geq \frac{2d}{d+2}$ is not necessary. For $s = 0$ it does not occur (cf. Remark 4.3) and for $s > 0$ one works with the space $X = W_0^{1,p}(\Omega) \cap L^2(\Omega)$. One can show that both X and the operator $A : X \to X^*$ fulfill the conditions of BrowderMinty's Theorem 2.1.*

(ii) Theorem 4.5 can also be applied to the equation

$$\begin{aligned} -\operatorname{div}(\mathbf{A}(x, \nabla u)) &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

if $\mathbf{A} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ fulfills the following conditions:

(a) A is a Carathéodory function,

(b) $|\mathbf{A}(x, \boldsymbol{\eta})| \leq C\left(g(x) + |\boldsymbol{\eta}|^{p-1}\right), g \in L^{p'}(\Omega)$ (growth condition),

(c) $(\mathbf{A}(x, \boldsymbol{\eta}) - \mathbf{A}(x, \boldsymbol{\zeta})) \cdot (\boldsymbol{\eta} - \boldsymbol{\zeta})) > 0$, for almost all x (strict monotonicity),

(d) $\mathbf{A}(x, \boldsymbol{\eta}) \cdot \boldsymbol{\eta} \geq c|\boldsymbol{\eta}|^p - h(x), h \in L^1(\Omega)$ (coercivity).

(iii) Theorem 4.55 also holds for any $f \in \left(W_0^{1,p}(\Omega)\right)^*$. It can be shown that such f is a representation of the form

$$f = \sum_{i=1}^{d} \partial_i f_i + f_0$$

with $f_i \in L^{p'}(\Omega), i = 0, \ldots, d$.

5 Pseudomonotone operators

The aim of this section is to develop a theory that makes it possible to solve quasilinear elliptic equations that contain a term of lower order that is not monotone. For example, the equation

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + su &= f \quad \text{in}\,\Omega, \\ u &= 0 \quad \text{on}\,\partial\Omega, \end{aligned}$$

cannot be solved using the theory of monotone operators if $s < 0$. However, an inspection of the proof of Browder-Minty's Theorem 2.1 shows that the arguments can be adapted to more general operators, namely pseudomonotone operators. Typical examples of pseudomonotone operators are operators of the form

$$A = A_1 + A_2,$$

where $A_1 : X \to X^*$ is a monotone, hemicontinuous operator and $A_2 : X \to X^*$ is a fully continuous, i.e. compact, operator (cf. Lemma 1.6 (i)), i.e. the theory of pseudomonotone operators combines monotonicity and compactness. In the following, we will first develop a general theory and then apply it to differential equations.

Definition 5.1. *Let X be a Banach space and $A : X \to X^*$ an operator. We say that A is of type M if from*

$$\begin{aligned} u_n &\rightharpoonup u && \text{in } X \quad (n \to \infty) \\ Au_n &\rightharpoonup b && \text{in } X^* \quad (n \to \infty) \\ \limsup_{n\to\infty} \langle Au_n, u_n\rangle_X &\leq \langle b, u\rangle_X \end{aligned}$$

it follows that $Au = b$ applies.

This condition is important because it is invariant under continuous perturbations. Moreover, monotone operators fulfill this condition.

Lemma 5.2. *Let X be a reflexive Banach space and let $A : X \to X^*$, $B : X \to X^*$ be operators. Then the following holds:*
(i) If A is monotone and hemicontinuous, then A is of type M.
(ii) If A is of type M and B is continuous, then $A + B$ is of type M.

Proof. ad (i): This is exactly the statement of Lemma 0.2 (ii).
ad (ii): Given a sequence $(u_n) \subseteq X$ with

$$\begin{aligned} u_n \rightharpoonup u \text{ in } X \quad (n \to \infty), \\ Au_n + Bu_n \rightharpoonup b \text{ in } X^* \quad (n \to \infty), \\ \limsup_{n\to\infty} \langle Au_n + Bu_n, u_n\rangle_X \leq \langle b, u\rangle_X. \end{aligned}$$

Since B is continuous, $Bu_n \to Bu$ in $X^*(n \to \infty)$ and thus

$$Au_n \rightharpoonup b - Bu \text{ in } X^* \quad (n \to \infty).$$
$$\limsup_{n\to\infty} \langle Au_n, u_n\rangle_X \leq \langle b - Bu, u\rangle_X$$

Since A is of type M, $Au = b - Bu$, i.e. $Au + Bu = b$. □

For operators $A : X \to X^*, B : X \to X^*$ that are of type M, $A + B$ is not necessarily of type M. Therefore, we introduce the more stable notion of pseudomonotone operator.

Definition 5.3. *Let $A : X \to X^*$ be an operator on a Banach space X. Then A is called pseudomonotone if from*

$$u_n \rightharpoonup u \quad \text{in } X \quad (n \to \infty),$$
$$\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle_X \leq 0$$

it follows that for all $w \in X$ applies:

$$\langle Au, u - w\rangle_X \leq \liminf_{n\to\infty} \langle Au_n, u_n - w\rangle_X$$

The following lemma gives typical examples of pseudomonotone operators and important properties.

Lemma 5.4. *Let X be a reflexive Banach space, and let $A, B : X \to X^*$ be operators. Then holds:*
(i) If A is monotone and hemicontinuous, then A is pseudomonotone.
(ii) If A is continuous, then A is pseudomonotone.
(iii) If A and B are pseudomonotone, then $A + B$ is pseudomonotone.
(iv) If A is pseudomonotone, then A is of type M.
(v) If A is pseudomonotone and locally bounded, then A is demicontinuous.

Proof. ad (i): Given a sequence $(u_n) \subseteq X$ with $u_n \rightharpoonup u$ in $X (n \to \infty)$ and

$$\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle_X \leq 0$$

Since A is monotone, the following applies:

$$\langle Au_n - Au, u_n - u\rangle_X \geq 0$$

from which follows

$$\liminf_{n\to\infty} \langle Au_n, u_n - u\rangle_X \geq \liminf_{n\to\infty} \langle Au, u_n - u\rangle_X = 0.$$

So together we get

$$\lim_{n\to\infty} \langle Au_n, u_n - u\rangle_X = 0$$

For any $w \in X$ and $t > 0$ we set $z_t := (1 - t)u + tw$. The monotonicity of A implies

$$\langle Au_n - Az_t, u_n - z_t\rangle_X \geq 0$$

which, due to the choice of z_t, is equivalent to

$$t \langle Au_n, u_n - w\rangle_X \geq -(1 - t) \langle Au_n, u_n - u\rangle_X + (1 - t) \langle Az_t, u_n - u\rangle_X + t \langle Az_t, u_n - w\rangle_X$$

is. Thus, for all $w \in X$ and $t > 0$ we get :

$$\liminf_{n\to\infty} \langle Au_n, u_n - w\rangle_X \geq \langle Az_t, u - w\rangle_X$$

where we used $u_n \to u$ in $X(n \to \infty)$, and $t > 0$. Since we can also write z_t as $z_t = u + t(w-u)$ and the operator A is hemicontinuous, we get $Az_t \to Au$ for $t \searrow 0^+$. Thus, for all $w \in X$:

$$\liminf_{n\to\infty} \langle Au_n, u_n - w\rangle_X \geq \langle Au, u-w\rangle_X$$

i.e. A is pseudomonotone.

ad (ii): Let $(u_n) \subseteq X$ be a sequence with $u_n \to u$ in $X(n \to \infty)$. Then $Au_n \to Au$ in $X^*(n \to \infty)$, due to the continuity of A. Using Lemma 0.3 (ii), we thus obtain for all $w \in X$

$$\langle Au, u-w\rangle_X = \lim_{n\to\infty} \langle Au_n, u_n - w\rangle_X$$

i.e. A is pseudomonotone.

ad (iii): We choose a sequence $(u_n) \subseteq X$ with $u_n \to u$ in $X(n \to \infty)$ and

$$\limsup_{n\to\infty} \langle Au_n + Bu_n, u_n - u\rangle_X \leq 0$$

From this follows

$$\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle_X \leq 0, \quad \limsup_{n\to\infty} \langle Bu_n, u_n - u\rangle_X \leq 0$$

which we prove by contradiction. So apply

$$\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle_X = a > 0$$

In particular, there is a subsequence (u_{n_k}) with

$$\lim_{k\to\infty} \langle Au_{n_k}, u_{n_k} - u\rangle_X = a$$

and thus we obtain

$$\begin{aligned}
&\limsup_{k\to\infty} \langle Bu_{n_k}, u_{n_k} - u\rangle_X \\
&\quad = \limsup_{k\to\infty} \langle (A+B)u_{n_k} - Au_{n_k}, u_{n_k} - u\rangle_X \\
&\quad \leq \limsup_{k\to\infty} \langle (A+B)u_{n_k}, u_{n_k} - u\rangle_X + \limsup_{k\to\infty} \langle -Au_{n_k}, u_{n_k} - u\rangle_X \\
&\quad = \limsup_{k\to\infty} \langle (A+B)u_{n_k}, u_{n_k} - u\rangle_X - \lim_{k\to\infty} \langle Au_{n_k}, u_{n_k} - u\rangle_X \\
&\quad \leq -a.
\end{aligned}$$

Since B is pseudomonotone, the following applies for all $w \in X$:

$$\langle Bu, u-w\rangle_X \leq \liminf_{k\to\infty} \langle Bu_{n_k}, u_{n_k} - w\rangle_X$$

For $w = u$ we therefore obtain

$$0 \leq \liminf_{k\to\infty} \langle Bu_{n_k}, u_{n_k} - u\rangle_X \leq \limsup_{k\to\infty} \langle Bu_{n_k}, u_{n_k} - u\rangle_X \leq -a < 0$$

which is a contradiction. Thus, with the pseudomonotoneity of A and B, it yields

$$\langle Au, u - w \rangle_X \leq \liminf_{n\to\infty} \langle Au_n, u_n - w \rangle_X ,$$
$$\langle Bu, u - w \rangle_X \leq \liminf_{n\to\infty} \langle Bu_n, u_n - w \rangle_X .$$

If we add both inequalities, the result for all $w \in X$ is

$$\langle Au + Bu, u - w \rangle_X \leq \liminf_{n\to\infty} \langle Au_n + Bu_n, u_n - w \rangle_X$$

i.e. $A + B$ is pseudomonotone.

ad (iv): Given a sequence $(u_n) \subseteq X$ that satisfies the pseduomonotone conditions. This implies in particular

$$\limsup_{n\to\infty} \langle Au_n, u_n - u \rangle_X \leq 0$$

Due to the pseudomonotoneity of A, we thus obtain for all $w \in X$

$$\begin{aligned}\langle Au, u - w \rangle_X &\leq \liminf_{n\to\infty} \langle Au_n, u_n - w \rangle_X \\ &\leq \langle b, u \rangle_X - \langle b, w \rangle_X = \langle b, u - w \rangle_X\end{aligned}$$

If we replace w with $2u - w$, the result for all $w \in X$ is :

$$\langle Au, u - w \rangle_X = \langle b, u - w \rangle_X$$

i.e. $Au = b$.

ad (v): Let $(u_n) \subseteq X$ be a sequence with $u_n \to u$ in $X (n \to \infty)$. Since A is locally bounded, the sequence (Au_n) is also bounded. The space X is reflexive and therefore there is a subsequence (Au_{n_k}) with $Au_{n_k} \to b$ in $X^*(k \to \infty)$, so that we get $\lim_{k\to\infty} \langle Au_{n_k}, u_{n_k} - u \rangle_X = 0$. The pseudomonotoneity of A together with the above convergences implies for all $w \in X$:

$$\begin{aligned}\langle Au, u - w \rangle_X &\leq \liminf_{k\to\infty} \langle Au_{n_k}, u_{n_k} - w \rangle_X \\ &= \langle b, u - w \rangle_X\end{aligned}$$

Thus, as in (iv), $Au = b$, i.e. $Au_{n_k} \to Au$ in $X^*(k \to \infty)$. The convergence principle Lemma 0.3 (iv) provides, since the above argument applies to any convergent subsequences,

$$Au_n \to b = Au \quad \text{in } X^* \quad (n \to \infty),$$

i.e. A is demicontinuous. □

Theorem 5.5 (Brezis)**.** *Let $A : X \to X^*$ be a pseudomonotone, bounded, coercive operator, where X is a separable, reflexive Banach space. Then for all $b \in X^*$ there exists a solution $u \in X$ of*

$$Au = b.$$

Proof. Due to Lemma 5.4 (v), A is demicontinuous, since A is pseudomonotone and bounded. According to Lemma 5.4 (iv), A also is of type M, since A is pseudomonotone. We now proceed analogously to the proof of Browder-Minty's Theorem 2.1. To do this, we choose a basis $(w_i)_{i\in\mathbb{N}}$ of X. Using the Galerkin method, we look for approximate solutions

$$u_n = \sum_{k=1}^{n} c_n^k w_k$$

which represent the Galerkin system (cf. proof of Browder-Minty's Theorem 2.1)

$$g_n^k(\mathbf{c}_n) = g_n^k(u_n) := \langle Au_n - b, w_k\rangle_X = 0, \quad k = 1, \dots, n$$

can be solved. The solvability of this system of equations follows as in the proof of Browder-Minty's Theorem 2.1, since A is demicontinuous and coercive. The demicontinuousness of A implies that the functions $g_n^k, k = 1, \dots, n$, are continuous, and the coercivity of A, that there is an $R_0 > 0$ such that for all $\|u_n\|_X = R_0$ $\sum_{k=1}^n g_n^k(\mathbf{c}_n)\, c_n^k > 0$. Thus, a corollary of Brouwer's Theorem provides the existence of a solution u_n of the Galerkin system, as well as the a priori estimate.

$$\|u_n\|_X \le R_0 \quad \forall n \in \mathbb{N}$$

So there is a convergent subsequence (u_{n_k}) with $u_{n_k} \to u$ in $X (k \to \infty)$. We now want to show that u solves $Au = b$. From the Galerkin system it follows that

$$\lim_{k\to\infty} \langle Au_{n_k}, v\rangle_X = \langle b, v\rangle_X \quad \forall v \in \bigcup_{n\in\mathbb{N}} \operatorname{span}(w_1, \dots, w_n)$$

The boundedness of the operator A provides that the sequence (Au_{n_k}) is bounded, since the weakly convergent sequence (u_{n_k}) is bounded. Due to the reflexivity of X^*, a subsequence of (Au_{n_k}), which we again denote by (Au_{n_k}), has a weak limit, i.e.

$$Au_{n_k} \to c \text{ in } X^* \quad (k \to \infty).$$

However, $c = b$ applies with the same arguments as in proof part 3 of Browder-Minty's Theorem 2.1. From the Galerkin system and the weak convergence of (u_{n_k}) we obtain

$$\langle Au_{n_k}, u_{n_k}\rangle_X = \langle b, u_{n_k}\rangle_X \to \langle b, u\rangle_X \quad (k \to \infty).$$

Therefore, the sequence (u_{n_k}) fulfills the requirements of the type M and it follows

$$Au = b,$$

i.e. u is the required solution. □

6 Quasilinear elliptic equations II

We now consider the problem

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + g(u) = f \qquad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \partial\Omega, \end{aligned}$$

where $\Omega \subseteq \mathbb{R}^d$ is a bounded domain with Lipschitz continuous boundary $\partial\Omega$, $f : \Omega \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are given functions and $u : \Omega \to \mathbb{R}$ is the function we are looking for. We restrict ourselves to the case $p < d$ in the following. However, all results also apply to the case $p \geq d$. We want to use the theory of pseudomonotone operators. To do this, we set $X = W_0^{1,p}(\Omega)$ and define the following mappings:

$$\begin{aligned} \langle A_1 u, \varphi \rangle_X &:= \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx \\ \langle A_2 u, \varphi \rangle_X &:= \int_\Omega g(u) \varphi dx, \\ \langle b, \varphi \rangle_X &:= \int_\Omega f \varphi dx. \end{aligned}$$

We proceed analogously to Section 4. The operator A_1 and the functional b have already been dealt with there. For the operator A_2 we apply:

Lemma 6.1. *Let Ω be a bounded domain of $\mathbb{R}^d$ with Lipschitz continuous boundary $\partial\Omega$. The continuous function $g : \mathbb{R} \to \mathbb{R}$ we impose the following growth condition:*

$$|g(t)| \leq c \left(1 + |t|^{r-1}\right)$$

where $1 \leq r < \infty$. For $1 \leq p < d$ and $r \leq \frac{dp}{d-p} =: q$ the operator A_2 forms the space $X = W_0^{1,p}(\Omega)$ into its dual space X^*ab and is bounded. For $r < \frac{dp}{d-p} = q$ A_2 is continuous.

Proof. (i) From the definition of A_2 we obtain for $u, \varphi \in X$

$$\begin{aligned} |\langle A_2 u, \varphi \rangle_X| &\leq \int_\Omega c \left(1 + |u|^{r-1}\right) |\varphi| dx \\ &\leq c \int_\Omega |\varphi| dx + c \left(\int_\Omega |u|^{(r-1)q'} dx \right)^{\frac{1}{q'}} \left(\int_\Omega |\varphi|^q dx \right)^{\frac{1}{q}}. \end{aligned}$$

If we use the Sobolev embedding $X = W_0^{1,p}(\Omega) \hookrightarrow L^\alpha(\Omega), \alpha \leq q$ and $(r-1)q' \leq q$, which is equivalent to $r \leq \frac{dp}{d-p}$ due to the definition of q, we obtain

$$\begin{aligned} \|A_2 u\|_{X^*} &= \sup_{\|\varphi\|_X \leq 1} |\langle A_2 u, \varphi \rangle_X| \\ &\leq \sup_{\|\varphi\|_X \leq 1} c \left(1 + \|u\|_X^{r-1}\right) \|\varphi\|_X \leq c \left(1 + \|u\|_X^{r-1}\right). \end{aligned}$$

Consequently, $A_2 u \in X^*$, i.e. $A_2 : X \to X^*$. It also follows from this estimate that A_2 is bounded.

(ii) Let $(u_n) \subseteq X$ be a weakly convergent sequence. Due to the compact embedding $X = W_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$, for $r < \frac{dp}{d-p}$, the following holds

$$u_n \to u \text{ in } L^r(\Omega) \quad (n \to \infty).$$

We set

$$(Fv)(x) := g(v(x))$$

and using the growth condition and the continuity of g we obtain that the Nemyckii operator F fulfills the conditions of Lemma 3.1. With $r-1 = \frac{r}{r'}$, $F : L^r(\Omega) \to L^{r'}(\Omega)$ is therefore continuous; particularly:

$$\|F(u_n) - F(u)\|_{r'} \to 0 \quad (n \to \infty)$$

From this we get

$$\begin{aligned}\sup_{\|\varphi\|_X \le 1} |\langle A_2 u_n - A_2 u, \varphi\rangle_X| &\le \sup_{\|\varphi\|_X \le 1} \int_\Omega |g(u_n) - g(u)\|\varphi| \, dx \\ &\le \sup_{\|\varphi\|_X \le 1} \|F(u_n) - F(u)\|_{r'} \|\varphi\|_r \\ &\le c \|F(u_n) - F(u)\|_{r'}\end{aligned}$$

due to the embedding $X \hookrightarrow L^r(\Omega)$. Further, we have $A_2 u_n \to A_2 u$ in $X^*(n \to \infty)$, i.e. A_2 is continuous.

□

In order to apply Brezis' theorem, we need the following lemma.

Lemma 6.2. *In addition to the conditions of Lemma 6.1, let g fulfill the coercivity condition*

$$\inf_{t \in \mathbb{R}} g(t)t > -\infty$$

and let $p > 1$. Then the operator $A_1 + A_2 : X \to X^*$ is coercive.

Proof. Before, it was shown that

$$\langle A_1 u, u\rangle_X = \|\nabla u\|_p^p$$

applies. Further, there exists a constant $c_0 > 0$ such that

$$\langle A_2 u, u\rangle_X = \int_\Omega g(u) u dx > -c_0.$$

Thus we obtain for $p > 1$

$$\frac{\langle A_1 u + A_2 u, u\rangle_X}{\|u\|_X} > \frac{\langle A_1 u, u\rangle_X}{\|\nabla u\|_p} - \frac{c_0}{\|\nabla u\|_p} = \|\nabla u\|_p^{p-1} - \frac{c_0}{\|\nabla u\|_p} \to \infty$$

if $\|\nabla u\|_p \to \infty$. So the operator $A_1 + A_2$ is coercive. □

Theorem 6.3. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with boundary $\partial\Omega \in C^{0,1}$. Let $1 < p < d$ and the continuous function $g : \mathbb{R} \to \mathbb{R}$ fulfills the conditions of Lemma 6.2 with $1 \le r < \frac{dp}{d-p}$. Then for all $f \in L^{p'}(\Omega)$ there exists a weak solution i.e. there is a $u \in X = W_0^{1,p}(\Omega)$ such that*

$$(A_1 + A_2) u = b$$

Proof. Let us apply Brezis' theorem. The space $X = W_0^{1,p}(\Omega)$ is a reflexive separable Banach space. We know that $A_1 : X \to X^*$ is a strictly monotone, continuous, bounded operator. Thus A_1 is pseudomonotone. Moreover, A_2 is a fully continuous, bounded operator. Therefore A_2 is pseudomonotone. Overall, $A_1 + A_2$ is a pseudomonotone operator, which is also coercive. The assertion now follows immediately from the theorem of Brezis. □

Remark 6.4. *(i) The case $p \geq d$ can be treated analogously. In this case, the upper bound for r is omitted, i.e. all $r \in [1, \infty)$ are admissible. However, case distinctions must be made in the embedding sets, which would have further complicated the above calculations. (ii) The function $g(t) = -\alpha t, \alpha > 0$, is not covered by the previous theorem, since g does not fulfill the coercivity condition. For $p \geq \frac{2d}{d+2}$ the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$ holds and thus the coercivity of $A_1 + A_2$ can be proved as follows:*

$$\begin{aligned}\frac{\langle A_1 u + A_2 u, u\rangle_X}{\|u\|_X} &= \|\nabla u\|_p^{p-1} - \alpha \frac{\|u\|_2^2}{\|\nabla u\|_p} \\ &\geq \|\nabla u\|_p^{p-1} - \alpha c_0 \|\nabla u\|_p\end{aligned}$$

where c_0 is the embedding constant of $W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$. The right-hand side tends to infinity if either $p > 2$ or $p = 2$ and $\alpha c_0 < 1$. Thus $A_1 + A_2$ is coercive and one can proceed as before, which provides the existence of generalized solutions for $p \geq 2$.

7 The stationary Navier-Stokes equations

The stationary Navier-Stokes equations are

$$\begin{aligned}-\Delta \mathbf{u} + [\nabla \mathbf{u}]\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega \\ \operatorname{div} \mathbf{u} &= 0 \qquad\qquad \text{in } \Omega \\ \mathbf{u} &= 0 \quad \text{on } \partial\Omega\end{aligned}$$

We use the notation $[\nabla \mathbf{u}]\mathbf{u} := \left(\sum_{j=1}^{3} u^j \left(\partial_j u^i\right)\right)_{i=1,2,3}$ where $\Omega \subseteq \mathbb{R}^3$ is a bounded domain with Lipschitz continuous boundary $\partial\Omega$. These equations describe the steady flow of a viscous, incompressible fluid. It is $\mathbf{u} = (u^1, u^2, u^3)^\top : \Omega \to \mathbb{R}^3$ the velocity, $p : \Omega \to \mathbb{R}$ the pressure and $\mathbf{f} : \Omega \to \mathbb{R}^3$ an external force. The term $[\nabla \mathbf{u}]\mathbf{u}$ is often called the vortex term. The pressure can only be determined from equations up to a constant. It is therefore possible to impose a further condition on p, whereby we opt for $\int_\Omega p dx = 0$ for the sake of simplicity. We set

$$X := \left\{\boldsymbol{\varphi} \in W_0^{1,2}\left(\Omega; \mathbb{R}^3\right) \mid \operatorname{div} \boldsymbol{\varphi} = 0\right\}$$

This is a linear subspace of $W_0^{1,2}(\Omega; \mathbb{R}^3)$, which we can describe with the norm

$$\|\mathbf{u}\|_X := \|\nabla \mathbf{u}\|_{L^2(\Omega; \mathbb{R}^{3\times 3})}$$

provided. We define for all $\mathbf{u}, \varphi \in X$ and $p \in L^2(\Omega)$ with $\int_\Omega p dx = 0$

$$\begin{aligned}\langle A_1 \mathbf{u}, \boldsymbol{\varphi}\rangle_X &:= \int_\Omega \nabla \mathbf{u} \cdot \nabla \boldsymbol{\varphi} dx \\ \langle A_2 \mathbf{u}, \boldsymbol{\varphi}\rangle_X &:= \int_\Omega [\nabla \mathbf{u}]\mathbf{u} \cdot \boldsymbol{\varphi} dx \\ \langle P, \boldsymbol{\varphi}\rangle_X := \langle \nabla p, \boldsymbol{\varphi}\rangle_X &:= -\int_\Omega p \operatorname{div} \boldsymbol{\varphi} dx = 0, \\ \langle b, \boldsymbol{\varphi}\rangle_X &:= \int_\Omega \mathbf{f} \cdot \boldsymbol{\varphi} dx.\end{aligned}$$

Obviously, the operator equation $A_1\mathbf{u} + A_2\mathbf{u} = b$ is equivalent to the weak formulation, i.e. for all $\varphi \in X$ the following applies

$$\int_\Omega \nabla\mathbf{u}\cdot\nabla\varphi dx+\int_\Omega[\nabla\mathbf{u}]\mathbf{u}\cdot\boldsymbol{\varphi}dx=\int_\Omega \mathbf{f}\cdot\varphi dx$$

We now check that the operators A_1, A_2 and the functional b are well-defined and that the operator A_1+A_2 satisfies the conditions of Brezis' theorem.

Lemma 7.1. *Under the above conditions on Ω, the space X is a reflexive separable Banach space.*

Proof. First we show that X is a closed subspace of $W_0^{1,2}(\Omega;\mathbb{R}^3)$. Let $(\mathbf{u}_n)\subseteq X$ be a sequence with $\mathbf{u}_n\to\mathbf{u}$ in $W_0^{1,2}(\Omega;\mathbb{R}^3)\,(n\to\infty)$. It follows in particular that $\nabla\mathbf{u}_n\to\nabla\mathbf{u}$ in $L^2(\Omega;\mathbb{R}^{3\times3})\,(n\to\infty)$. Therefore, there is a subsequence with $\nabla\mathbf{u}_{n_k}\to\nabla\mathbf{u}$ almost everywhere $(k\to\infty)$. So we get for almost all $x\in\Omega$

$$\operatorname{div}\mathbf{u}(x)=\operatorname{tr}\nabla\mathbf{u}(x)=\lim_{k\to\infty}\operatorname{tr}\nabla\mathbf{u}_{n_k}(x)=0,$$

i.e. $\mathbf{u}\in X$. Since a closed subspace of a Banach space is again a Banach space, we have proved that X is a Banach space. Moreover, a closed subspace of a reflexive Banach space is again reflexive. Since $W_0^{1,2}(\Omega;\mathbb{R}^3)$ is separable, the subspace $X\subseteq W_0^{1,2}(\Omega;\mathbb{R}^3)$ is also separable. □

Lemma 7.2. *Under the above conditions on Ω and X, the operator $A_1: X\to X^*$ is linear, continuous, coercive, strictly monotone and bounded.*

Proof. Obviously A_1 is linear. The operator $A_1: X\to X^*$ is a vector-valued variant of the operator $A: W_0^{1,2}(\Omega)\to\left(W_0^{1,2}(\Omega)\right)^*$ of the p-Laplace operator from before with $p=2$ and $s=0$. We can now adapt the proofs to our situation and obtain the missing assertions. □

Lemma 7.3. *The operator A_2 is a completely continuous and bounded operator from X to X^*.*

Proof. Firstly, for all $\mathbf{u},\varphi\in X$ it holds:

$$\begin{aligned}|\langle A_2\mathbf{u},\boldsymbol{\varphi}\rangle_X| &\le \int_\Omega |\mathbf{u}||\nabla\mathbf{u}||\boldsymbol{\varphi}|dx\\ &\le\left(\int_\Omega|\mathbf{u}|^4dx\right)^{\frac14}\left(\int_\Omega|\boldsymbol{\varphi}|^4dx\right)^{\frac14}\left(\int_\Omega|\nabla\mathbf{u}|^2dx\right)^{\frac12}\\ &\le c\|\nabla\mathbf{u}\|_2^2\|\boldsymbol{\varphi}\|_X,\end{aligned}$$

because $X\hookrightarrow L^4(\Omega;\mathbb{R}^3)$. From the estimation follows both $A_2\mathbf{u}\in X^*$ and thus $A_2: X\to X^*$, as well as the boundedness of A_2.

Next, we prove that A_2 is completely continuous: Let $(\mathbf{u}_n)\subseteq X$ be a sequence with $\mathbf{u}_n\rightharpoonup\mathbf{u}$ in $X(n\to\infty)$. From the compact embedding $X\hookrightarrow\hookrightarrow L^4(\Omega;\mathbb{R}^3)$ follows $\mathbf{u}_n\to\mathbf{u}$ in $L^4(\Omega;\mathbb{R}^3)$ $(n\to\infty)$. For all $\varphi\in X$ applies:

$$\begin{aligned}|\langle A_2\mathbf{u}_n-A_2\mathbf{u},\boldsymbol{\varphi}\rangle_X| &=\left|\int_\Omega[\nabla\mathbf{u}_n]\,\mathbf{u}_n\cdot\boldsymbol{\varphi}-[\nabla\mathbf{u}]\mathbf{u}\cdot\boldsymbol{\varphi}dx\right|\\ &=\left|\int_\Omega[\nabla\mathbf{u}_n]\,(\mathbf{u}_n-\mathbf{u})\cdot\boldsymbol{\varphi}+[\nabla(\mathbf{u}_n-\mathbf{u})]\,\mathbf{u}\cdot\boldsymbol{\varphi}dx\right|\\ &=\left|\int_\Omega[\nabla\mathbf{u}_n]\,(\mathbf{u}_n-\mathbf{u})\cdot\boldsymbol{\varphi}+[\nabla\boldsymbol{\varphi}]\mathbf{u}\cdot(\mathbf{u}_n-\mathbf{u})\,dx\right|\end{aligned}$$

where we have partially integrated in the last step and utilized div $\mathbf{u} = 0$. Thus, using the Hölder inequality, we obtain

$$\begin{aligned}\|A_2\mathbf{u}_n - A_2\mathbf{u}\|_{X^*} &= \sup_{\|\boldsymbol{\varphi}\|_X \le 1} |\langle A_2\mathbf{u}_n - A_2\mathbf{u}, \boldsymbol{\varphi}\rangle_X| \\ &\le \sup_{\|\boldsymbol{\varphi}\|_X \le 1} \|\mathbf{u}_n - \mathbf{u}\|_4 \|\nabla \mathbf{u}_n\|_2 \|\boldsymbol{\varphi}\|_4 + \|\mathbf{u}\|_4 \|\mathbf{u}_n - \mathbf{u}\|_4 \|\nabla \boldsymbol{\varphi}\|_2 \\ &\le c\|\mathbf{u}_n - \mathbf{u}\|_4 \|\nabla \mathbf{u}_n\|_2 + \|\mathbf{u}\|_4 \|\mathbf{u}_n - \mathbf{u}\|_4 \to 0 \quad (n \to \infty),\end{aligned}$$

where we used the embedding $X \hookrightarrow L^4(\Omega;\mathbb{R}^3)$, as well as the boundedness of the sequence $(\|\nabla \mathbf{u}_n\|_2\ )$ (cf. Lemma 0.3 (i)) and the convergence $\mathbf{u}_n \to \mathbf{u}$ in $L^4(\Omega;\mathbb{R}^3)$ $(n \to \infty)$. Thus A_2 is completely continuous on X. □

Theorem 7.4. *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with Lipschitz continuous boundary $\partial\Omega$. Then for every $\mathbf{f} \in L^2(\Omega;\mathbb{R}^3)$ there is a $\mathbf{u} \in X$ so that $\mathbf{u}$ solves the Navier-Stokes equations in the weak sense.*

Proof. The space X is a separable and reflexive Banach space. Moreover, the operator $A_1 + A_2 : X \to X^*$ is bounded and pseudomonotone. It remains to show that $A_1 + A_2$ is also coercive. For all $\mathbf{u} \in X$ we have:

$$\begin{aligned}\langle A_2\mathbf{u}, \mathbf{u}\rangle_X &= \int_\Omega \sum_{i,j=1}^3 u^j \left(\partial_j u^i\right) u^i dx = \frac{1}{2}\int_\omega \sum_{j=1}^3 u^j \partial_j |\mathbf{u}|^2 dx \\ &= -\frac{1}{2}\int_\Omega \operatorname{div} \mathbf{u}|\mathbf{u}|^2 dx = 0\end{aligned}$$

since for $\mathbf{u} \in X$ holds $\operatorname{div} \mathbf{u} = 0$. Since A_1 is coercive, $A_1 + A_2$ is coercive on X^3. Brezis' theorem provides the assertion of the theorem. □

We note that coercivity is the only property that only applies to X and not to $W_0^{1,2}(\Omega;\mathbb{R}^3)$. To prove the other properties, we do not need the condition div $\mathbf{u} = 0$.
So far, we have shown the existence of a velocity u that satisfies the weak form for any $\varphi \in X$. To also find a pressure p such that for all $\varphi \in W_0^{1,2}(\Omega;\mathbb{R}^3)$

$$\int_\Omega \nabla\mathbf{u} \cdot \nabla\varphi dx + \int_\Omega [\nabla\mathbf{u}]\mathbf{u} \cdot \varphi dx + \int_\Omega p \operatorname{div} \varphi dx = \int_\Omega \mathbf{f} \cdot \varphi dx$$

applies, one must apply De Rham's theorem to $\left(\mathbf{F} \in W_0^{1,2}(\Omega;\mathbb{R}^3)\right)^*$, defined by

$$\langle \mathbf{F}, \boldsymbol{\varphi}\rangle_{W_0^{1,2}(\Omega;\mathbb{R}^3)} := \int_\Omega \nabla\mathbf{u} \cdot \nabla\boldsymbol{\varphi} dx + \int_\Omega [\nabla\mathbf{u}]\mathbf{u} \cdot \boldsymbol{\varphi} dx - \int_\Omega \mathbf{f} \cdot \boldsymbol{\varphi} dx$$

Theorem 7.5 (De Rham)**.** . *Let $\mathbf{F} \in \left(W_0^{1,2}(\Omega;\mathbb{R}^3)\right)^*$ a functional. If the following applies for all $\varphi \in X$:*

$$\langle \mathbf{F}, \boldsymbol{\varphi}\rangle_{W_0^{1,2}(\Omega;\mathbb{R}^3)} = 0$$

then there exists a function $p \in L^2(\Omega)$ with $\int_\Omega p dx = 0$, so that for all $\varphi \in W_0^{1,2}(\Omega;\mathbb{R}^3)$ holds:

$$\langle \mathbf{F}, \boldsymbol{\varphi}\rangle_{W_0^{1,2}(\Omega;\mathbb{R}^3)} = \int_\Omega p \operatorname{div} \varphi dx$$

Proof. See exercises. □

8 Evolution problems

Before we deal with the nonstationary versions of the p-Laplace problem, let us consider some special features of the treatment of time-dependent problems. The first peculiarity is that in the study of parabolic differential equations and evolution equations, the location and time variables are treated differently. What does this mean? In equations of this type, the unknown is a function u that lies in a function space X whose elements are defined on the space-time cylinder $I \times \Omega$, where Ω is a bounded domain in $\mathbb{R}^d$ and $I = (0, T)$ is a given time interval. Now each $u : I \times \Omega \to \mathbb{R}$ can be given by the rule

$$[\tilde{u}(t)](x) := u(t, x)$$

a mapping $\tilde{u} : I \to Y$, where Y is a function space whose elements are only defined on Ω. Thus, for all $t \in I$ we can assign the function $\tilde{u}(t) : \Omega \to \mathbb{R} : x \mapsto u(t, x)$ as an element of this function space. This gives us two views of u : On the one hand, u can be viewed as a function in time and space and, on the other hand, as a function in time with values in a location-dependent function space. In the following we will use the second view. Another special feature of the treatment of parabolic differential equations is that several function spaces occur naturally. We will illustrate this using the example of the heat conduction equation

$$\begin{aligned} \partial_t u - \Delta u &= f && \text{in } I \times \Omega, \\ u &= 0 && \text{on } I \times \partial\Omega, \\ u(0) &= u_0 && \text{in } \Omega, \end{aligned}$$

to illustrate. Let u be a smooth solution, which of course also satisfies the weak formulation, i.e. for all $\varphi \in L^2\left(I; W_0^{1,2}(\Omega)\right)$:

$$\int_I \int_\Omega \partial_t u \varphi dx dt + \int_I \int_\Omega \nabla u \cdot \nabla \varphi dx dt = \int_I \int_\Omega f \varphi dx dt$$

If we now choose $\varphi = u$, we obtain the a priori estimate using partial integration and Oneng's inequality

$$\|u\|^2_{L^\infty(I;L^2(\Omega))} + \|u\|^2_{L^2\left(I;W_0^{1,2}(\Omega)\right)} \le c\left(\|u_0\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(I;L^2(\Omega))}\right).$$

Using this estimate, we obtain the estimate

$$\|\partial_t u\|^2_{L^2\left(I;\left(W_0^{1,2}(\Omega)\right)^*\right)} \le c\left(\|u_0\|^2_{L^2(\Omega)}, \|f\|^2_{L^2(I;L^2(\Omega))}\right)$$

So, to treat the heat conduction equation in a natural way, we need the spaces $W_0^{1,2}(\Omega)$, $L^2(\Omega)$ and $W^{-1,2}(\Omega) := \left(W_0^{1,2}(\Omega)\right)^*$. We have seen that the time derivative $\partial_t u$ lies in the space $L^2\left(I; W^{-1,2}(\Omega)\right)$. Therefore, we want to take a closer look at how the time derivative is to be understood and develop a special case of the theory of generalized time derivatives.

Let V be a Banach space that embeds continuously in a Hilbert space H, i.e. there exists a linear, continuous and injective operator $j : V \to H$. We further assume that the embedding $V \overset{j}{\hookrightarrow} H$ is dense, i.e. $j(V) \subseteq H$ is dense in H. Under these conditions, we call (V, H, j) a Gelfand triple. A typical example is $\left(W_0^{1,2}(\Omega), L^2(\Omega), \mathrm{id}_{W_0^{1,2}(\Omega)}\right)$, i.e. exactly the spaces that occur naturally in the treatment of the heat conduction equation. The adjoint operator $j^* \in L\left(H^*, V^*\right)$, defined by

$$\langle j^* f, v\rangle_V := \langle f, jv\rangle_H \quad f \in H^*, v \in V,$$

i.e. the restriction of a continuous, linear functional $f \in H^*$ to the image domain $j(V)$, is again an embedding, because j^* is injective. In fact, from $j^* f = 0$ follows $0 = \langle j^* f, v\rangle_V = \langle f, jv\rangle_H$ for all $v \in V$ and thus $f = 0$ in H^*, since $j(V)$ is dense in H. If V is also reflexive, it can be shown that $j^*(H^*)$ is dense in V^* (exercise).
Due to Riesz's representation theorem, we can identify H with H^* using the Riesz operator $R : H \to H^*$, defined by

$$\langle Ry, x\rangle_H := (y, x)_H$$

can be identified. From the properties of the operators j, j^* and R it follows that $e := j^* \circ R \circ j : V \to V^*$, i.e.

$$V \overset{j}{\hookrightarrow} H \overset{R}{\cong} H^* \overset{j^*}{\hookrightarrow} V^*,$$

is continuous, linear and injective, i.e. an embedding from V to V^*. We call this canonical embedding of the Gelfand triple (V, H, j). With the help of the mapping e can be used to identify elements of the space V with elements of the dual space V^*. In the literature, the explicit mention of the embedding e is usually omitted. The same applies to the identifications using the mappings j and j^*. For all $v, w \in V$ the following applies

$$\langle ev, w\rangle_V = \langle j^* \circ R \circ jv, w\rangle_V = \langle Rjv, jw\rangle_H = (jv, jw)_H,$$

which together with the symmetry of the scalar product in H,

$$\langle ev, w\rangle_V = (jv, jw)_H = (jw, jv)_H = \langle ew, v\rangle_V$$

provides. Now we have all the tools together to define how we want to understand time derivatives.

Definition 8.1. *Let (V, H, j) be a Gelfand triple and let $1 < p < \infty$. Then a function $u \in L^p(I; V)$ has a generalized time derivative with respect to of the canonical embedding e, if there exists an element w of the space $L^{p'}(I; V^*)$, $\frac{1}{p'} + \frac{1}{p} = 1$, so that for all $v \in V$ and all $\varphi \in C_0^\infty(I; \mathbb{R})$*

$$\int_I \langle w(t), v\rangle_V \varphi(t) dt = -\int_I (ju(t), jv)_H \varphi'(t) dt$$

applies. If $u \in L^p(I; V)$ has a generalized time derivative, this is unique and we set $\partial_t(\boldsymbol{e}u) := w$.

Remark 8.2. *(i) The definition is closely related to the theory of distributions with values in Banach spaces. Let X be a Banach space and $q \in (1, \infty)$. Each element $f \in L^q(I; X)$ is defined by*

$$T_f(\varphi) := \int_I f(t)\varphi(t) dt, \quad \varphi \in C_0^\infty(I)$$

an element T_f of the space $\mathcal{D}'(I; X)$, the space of distributions with values in X. The derivative of the distribution T_f is the distribution $(T_f)' \in \mathcal{D}'(I; X)$ for which

$$(T_f)'(\varphi) = -T_f(\varphi')$$

applies to all $\varphi \in C_0^\infty(I)$. *If* $(T_f)'$ *can be represented by a function* $g \in L^q(I;X)$, *i.e.* $(T_f)' = T_g$, *we set* $f' := g$. *The Bochner–Sobolev space* $W^{1,q}(I;X)$ *consists of exactly such functions, i.e.*

$$W^{1,q}(I;X) := \{f \in L^q(I;X) \mid f' \in L^q(I;X)\}.$$

Analogous to the proof of the embedding $W^{1,p}(I) \hookrightarrow C(\bar{I})$, *one can show that every function* $f \in W^{1,q}(I;X)$ *has a representative* $f_c \in C(\bar{I};X)$ *and that the main theorem of differential and integral calculus*
integral calculus applies, i.e. for all $f \in W^{1,q}(I;X)$ *and all* $s, t \in \bar{I}$ *in* X *:*

$$f_c(t) = f_c(s) + \int_s^t f'(\tau)d\tau$$

(ii) Let (V, H, j) *be a Gelfand triple. The embeddings* $j : V \to H$ *and* $e : V \to V^*$ *define induced embeddings on Bochner spaces* $\boldsymbol{j} : L^p(I;V) \to L^p(I;H) : u \mapsto \boldsymbol{j}u$ *and* $\boldsymbol{e} : L^p(I;V) \to L^p(I;V^*) : u \mapsto \boldsymbol{e}u$, *where the induced functions* $\boldsymbol{j}u$ *and* $\boldsymbol{e}u$ *are defined by*

$$(\boldsymbol{j}u)(t) := j(u(t)), \quad (\boldsymbol{e}u)(t) := e(u(t)), \quad \text{for almost all } t \in I,$$

are defined. One can show that the functions $\boldsymbol{j}u$ *and* $\boldsymbol{e}u$ *are Bochner-measurable and that the operators* $\boldsymbol{j}$ *and* $\boldsymbol{e}$ *are continuous and linear. The injectivity of* $\boldsymbol{j}$ *and* $\boldsymbol{e}$ *follows immediately from the injectivity of* j *and* e. *Using (2.38), we see that the generalized time derivative* $\partial_t(\boldsymbol{e}u)$ *of a function* $u \in L^p(I;V)$ *is nothing other than the derivative of the distribution* T_{eu} *given by the function* $\partial_t(\boldsymbol{e}u) \in L^{p'}(I;V^*)$, *i.e.* $(T_{\boldsymbol{e}u})' = T_{\partial_t(\boldsymbol{e}u)}$ *in* $\mathcal{D}'(I;V^*)$.
(iii) In general, the generalized time derivative $\partial_t(\boldsymbol{e}u)$ *is not identical to the weak time derivative* $\partial_t u$. *A function* $u \in L^1(I \times \Omega)$ *has a weak time derivative* $\partial_t u \in L^1(I \times \Omega)$ *if*

$$\int_I \int_\Omega \partial_t u \varphi dx dt = -\int_I \int_\Omega u \partial_t \varphi dx dt \quad \forall \varphi \in C_0^\infty(I \times \Omega).$$

Both derivation concepts are therefore only comparable if $C_0^\infty(\Omega)$ *is dense in* V.

In the following we denote for a Gelfand triple $(V, H, j), p \in (1, \infty)$ and $I = (0, T)$

$$W^{1,p,p'}(I;V,V^*) = W := \left\{u \in L^p(I;V) \mid \partial_t(\boldsymbol{e}u) \in L^{p'}(I;V^*)\right\},$$
$$\|u\|_W := \|u\|_{L^p(I;V)} + \|\partial_t(\boldsymbol{e}u)\|_{L^{p'}(I;V^*)},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. The space $(W, \|\cdot\|_W)$ is a Banach space. It is reflexive if $1 < p < \infty$ and V is reflexive.

Lemma 8.3. *Let* V *be a reflexive separable Banach space,* (V, H, j) *a Gelfand triple.*
(i) For each function $u \in W$, *the function* $\boldsymbol{j}u \in L^p(I;H)$ *has a unique representative* $v \in C(\bar{I};H)$. *The resulting operator* $\boldsymbol{j} : W \to C(\bar{I};H) : u \mapsto v$ *is an embedding.*
(ii) For all $u, v \in W$ *and all* $s, t \in \bar{I}$ *holds:*

$$\int_s^t \langle \partial_t(\boldsymbol{e}u)(\tau), v(\tau)\rangle_V + \left\langle \frac{d_e v}{dt}(\tau), u(\tau)\right\rangle_V d\tau$$
$$= (\boldsymbol{j}u(t), \boldsymbol{j}v(t))_H - (\boldsymbol{j}u(s), \boldsymbol{j}v(s))_H.$$

Proof. Exercises. □

Remark 8.4. *(i) The formula is the analog of the following partial integration formula for real-valued functions $u, v : I \to \mathbb{R}$:*

$$\int_s^t u'(\tau)v(\tau) + u(\tau)v'(\tau)d\tau = u(t)v(t) - u(s)v(s)$$

(ii) In the special case $u = v \in W$ we get

$$\int_s^t \langle \partial_t(\boldsymbol{e}u)(\tau), u(\tau)\rangle_V \, d\tau = \frac{1}{2}\left\|\boldsymbol{j}u(t)\right\|_H^2 - \frac{1}{2}\left\|\boldsymbol{j}u(s)\right\|_H^2$$

We now consider the initial value problem

$$\begin{aligned} \partial_t(\boldsymbol{e}u) + Au &= b, \\ u(0) &= u_0 \end{aligned}$$

This is the transient variant of the operator equation $Au = b$. We are looking for solutions in a suitable Bochner space, and consequently the operator A is to be interpreted in a suitable way as an operator on such spaces. In the following, we restrict ourselves to the case where an operator $A : V \to V^*$ is given on a reflexive separable Banach space V which contains an induced operator $\mathcal{A}$ by the rule

$$\langle \mathcal{A}u, \varphi\rangle_{L^p(I;V)} := \int_I \langle A(u(t)), \varphi(t)\rangle_V dt \quad u, \varphi \in L^p(I;V)$$

is defined. Under certain conditions on A and p, many properties of the operator A are inherited by the induced operator $\mathcal{A}$.
The following theorem is a transient variant of Brezis' theorem.

Theorem 8.5. *Let V be a separable, reflexive Banach space and (V, H, j) a Gelfand triple, $p \in (1, \infty)$ and $I = (0, T)$ with $0 < T < \infty$. Let $A : V \to V^*$ be an operator such that the induced operator $\mathcal{A} : L^p(I;V) \to (L^p(I;V))^*$ is pseudomonotone and bounded, and the coercivity condition*

$$\langle \mathcal{A}u, u\rangle_{L^p(I;V)} \geq c_0\|u\|_{L^p(I;V)}^p, \quad u \in L^p(I;V), c_0 > 0$$

is sufficient. Then for all $u_0 \in H$ and all $b \in L^{p'}(I;V^*)$ there exists a solution $u \in W$, i.e. $u \in W$ satisfies $\boldsymbol{j}u(0) = u_0$ in H and for all $\varphi \in L^p(I;V)$ holds

$$\int_I \langle \partial_t(\boldsymbol{e}u)(t) + A(u(t)), \varphi(t)\rangle_V \, dt = \int_I \langle b(t), \varphi(t)\rangle_V dt$$

Proof. It is essentially the same as the proof of the more general result in Theorem ... below. □

In order to apply this theorem, we need a condition that ensures that the induced operator $\mathcal{A}$ of an operator $A : V \to V^*$ maps the space $L^p(I;V)$ into the dual space $(L^p(I;V))^*$.

Lemma 8.6. *Let V be a separable reflexive Banach space and $A : V \to V^*$ an operator.*
(i) If A is demicontinuous, the function defined for $u \in \mathcal{M}(I;V)$ by $t \mapsto A(u(t))$, for almost all $t \in I$, belongs to the space $\mathcal{M}(I;V^*)$.
(ii) If A is demistiguous and the growth condition

$$\|Au\|_{V^*} \leq c\left(\|u\|_V^{p-1} + 1\right), \quad u \in V, c > 0$$

suffices for a $p > 1$, the induced operator $\mathcal{A}$ maps the space $L^p(I;V)$ into the dual space $(L^p(I;V))^*$ and is bounded.

Proof. This lemma is a special case of the more general result in ... below. □

Before, we have treated the equation $(A_1 + A_2)u = b$ using Brezis' theorem on pseudomonotone operators, since it contains a monotone term and a compact term. Therefore, it makes sense to apply the theorem above to the transient version of this equation

$$\begin{aligned} \partial_t u - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + g(u) &= f \quad \text{in } I \times \Omega, \\ u &= 0 \quad \text{on } I \times \partial\Omega, \\ u(0) &= u_0 \quad \text{in } \Omega, \end{aligned}$$

to be applied. Let $p \in (1, \infty), \Omega$ be a bounded domain in $\mathbb{R}^d$ with $\partial\Omega \in C^{0,1}$ and $I = (0,T)$ a finite time interval. The right-hand side f and the initial value u_0 are given. We set $V := W_0^{1,p}(\Omega)$ and $H := L^2(\Omega)$, where we provide V with the equivalent norm $\|\nabla \cdot\|_p$. For $p \geq \frac{2d}{d+2}$ (V, H, id_V) is a Gelfand triple. We use the same operators as in the treatment of the quasilinear elliptic equation, i.e. we define

$$\begin{aligned} \langle A_1 u, v\rangle_V &:= \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v dx \\ \langle A_2 u, v\rangle_V &:= \int_\Omega g(u) v dx \end{aligned}$$

We discussed these operators before. In particular, we have shown that the operator $A_1 : V \to V^*$ for $p \geq \frac{2d}{d+2}$ is bounded, coercive, continuous and strictly monotone. These properties carry over to the induced operator $\mathcal{A}_1 : L^p(I;V) \to (L^p(I;V))^*$.

Lemma 8.7. *Let $p \geq \frac{2d}{d+2}$ and let $A_1 : V \to V^*$ defined as before. Then the induced operator $\mathcal{A}_1$ forms the space $L^p(I;V)$ into its dual space $(L^p(I;V))^*$ is bounded, continuous, strictly monotone and satisfies the coercivity condition*

$$\langle \mathcal{A}_1 u, u\rangle_{L^p(I;V)} \geq c_0 \|u\|^p_{L^p(I;V)}, \quad u \in L^p(I;V), c_0 > 0$$

In particular, $\mathcal{A}_1$ is pseudomonotone.

Proof. The operator A_1 is nothing else than the operator A from before for $s = 0$. We have shown that the operator $A_1 : V \to V^*$ satisfies the growth condition and we proved that $A_1 : V \to V^*$ is continuous. Thus, it follows that the induced operator $\mathcal{A}_1 : L^p(I;V) \to (L^p(I;V))^*$ is bounded. Just as before, we show that $u \in L^p(I;V)$ holds if and only if $u, \nabla u \in L^p(I \times \Omega)$. Thus, one can show that $\mathcal{A}_1$ is continuous, strictly monotone and coercive, but instead of Ω one has to work with $I \times \Omega$. In particular, $\mathcal{A}_1$ is pseudomonotone. □

The operator A_2 was treated before. There it was shown that $A_2 : V \to V^*$ is bounded and continuous if $r < \frac{dp}{d-p}$. The question now is whether these properties are also transferred to the induced operator $\mathcal{A}_2$, i.e. whether the operator $\mathcal{A}_2$ maps the space $L^p(I;V)$ into its dual space $(L^p(I;V))^*$ and is bounded and continuous there. We have shown that

$$\|A_2 u\|_{V^*} \leq c\left(1 + \|u\|_V^{r-1}\right)$$

applies if $g : \mathbb{R} \to \mathbb{R}$ is a continuous function with $r - 1$ growth. One can show that the induced

operator $\mathcal{A}_2$ thus maps the space $L^p(I;V)$ into $\left(L^{\left(\frac{p}{r-1}\right)'}(I;V)\right)^*$ and is bounded. Thus, if the rather restrictive condition $r \leq p$ is satisfied, $\mathcal{A}_2$ maps the space $L^p(I;V)$ into its dual space $(L^p(I;V))^*$.
The compact embedding $V = W_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega), q < \frac{dp}{d-p}$, was used to prove the full continuity of A_2. In general, however, the embedding

$$L^p(I;V) \hookrightarrow L^p\left(I;L^q(\Omega)\right),$$

with $q < \frac{dp}{d-p}$, is not compact. One can see this immediately if one takes a sequence $f_n : I \to \mathbb{R}$ which converges weakly in $L^p(I)$ against an $f \in L^p(I)$, but for which not $f_n \to f$ strongly holds in $L^p(I)(n \to \infty)$. For an arbitrary, but fixed, $v \in V$, the sequence

$$u_n(t,x) = f_n(t)v(x) \in L^p(I;V)$$

does not converge strongly to $L^p\left(I;L^q(\Omega)\right)$. In fact, the following applies

$$\begin{aligned}\|u_n - u\|^p_{L^p(I;L^q(\Omega))} &= \int_I \left(\int_\Omega |f_n(t) - f(t)|^q\, |v(x)|^q dx\right)^{\frac{p}{q}} dt \\ &= \|v\|^p_{L^q(\Omega)}\, \|f_n - f\|^p_{L^p(I)}\end{aligned}$$

and thus $u_n \to u$ in $L^p\left(I;L^q(\Omega)\right)(n \to \infty)$ converges if and only if $f_n \to f$ in $L^p(I)(n \to \infty)$. Thus, even for $r \leq p$, we do not know whether $\mathcal{A}_2 : L^p(I;V) \to (L^p(I;V))^*$ is continuous.
Thus, the transient existence theorem cannot be applied to the transient equation, since we do not know whether $\mathcal{A}_2 : L^p(I;V) \to (L^p(I;V))^*$ is continuous and therefore we do not know whether $\mathcal{A}_1 + \mathcal{A}_2 : L^p(I;V) \to (L^p(I;V))^*$ is pseudomonotone. One can construct simple examples that show that the operator $\mathcal{A}_1 + \mathcal{A}_2$ is not pseudomonotone. The above counterexample is based on the fact that no information about the time derivative was used, but only the information about the place derivative. In the existence theorem, we have seen that the solution u of the evolution problem lies in the space W and thus we have control over $\partial_t(\boldsymbol{e}u)$ in $L^{p'}(I;V^*)$. It can be shown that the restriction of the operator $\mathcal{A}_2$ to the space W, i.e. we consider $\mathcal{A}_2$ as an operator from W to W^*, is fully continuous. This means that $\mathcal{A}_1 + \mathcal{A}_2 : W \to W^*$ is pseudomonotone. However, even with this new information, we cannot apply the existence theorem, since we would have to work in the subspace W of $L^p(I;V)$ with a different topology and this situation is not dealt with in the theorem. In particular, we have no coercivity of the operator with respect to the space W.
In fact, the situation just discussed is prototypical in the application of the existence theorem. However, one can adapt the proof ideas in many concrete applications if one takes advantage of the fact that one also has control over the time derivative. This approach goes back to J.L. Lions and provides the existence of a solution to the problem. However, it introduces new technical complications, so that one has to consider a somewhat more general situation than the space W. Let us now briefly state the core result.
Let B, B_0, B_1 be Banach spaces, where B_0 and B_1 are reflexive and the following embeddings hold:

$$B_0 \overset{b_0}{\hookrightarrow} B \overset{b_1}{\hookrightarrow} B_1$$

i.e. in particular the embedding $b := b_1 \circ b_0 : B_0 \to B_1$ is compact. We set

$$W_0 := \{u \in L^{p_0}(I; B_0) \mid \partial_t(\boldsymbol{b}u) \in L^{p_1}(I; B_1)\}$$

where $1 < p_0, p_1 < \infty$, and provide W_0 with the norm

$$\|u\|_{W_0} := \|u\|_{L^{p_0}(I;B_0)} + \|\partial_t(\boldsymbol{b}u)\|_{L^{p_1}(I;B_1)}.$$

The embedding $b : B_0 \to B_1$ induces for each function $u \in L^{p_0}(I; B_0)$ a function $\boldsymbol{b}u : I \to B_1$. The generalized time derivative $\partial_t(\boldsymbol{b}u) \in L^{p_1}(I; B_1)$ of a function $u \in L^{p_0}(I; B_0)$ is to be understood in the sense that for all $\varphi \in C_0^\infty(I)$ applies in B_1 :

$$\int_I \boldsymbol{b}u(t)\varphi'(t)dt = -\int_I \frac{d_b u(t)}{dt}\varphi(t)dt$$

Obviously, W_0 is a reflexive Banach space and it holds:

$$W_0 \overset{\boldsymbol{b}_0}{\hookrightarrow} L^{p_0}(I; B).$$

However, one can prove the following stronger embedding:

Theorem 8.8 (Aubin-Lions)**.** *Under the conditions above and $1 < p_0, p_1 < \infty$ the following embedding is compact:*

$$W_0 \overset{\boldsymbol{b}_0}{\hookrightarrow} L^{p_0}(I; B).$$

The proof of this theorem is based on the following result.

Lemma 8.9 (Ehrling)**.** *Under the conditions above there is a constant $d(\eta)$ for all $\eta > 0$, so that for all $v \in B_0$ holds:*

$$\|b_0 v\|_B \leq \eta \|v\|_{B_0} + d(\eta)\|bv\|_{B_1}.$$

Proof. Assume that the inequality does not hold, then there is a $\eta > 0$ and sequences $(v_n) \subseteq B_0$ and $(d_n) \subseteq \mathbb{R}$ with $0 \leq d_n \to \infty (n \to \infty)$, so that

$$\|b_0 v_n\|_B > \eta \|v_n\|_{B_0} + d_n \|bv_n\|_{B_1}.$$

We set $w_n := v_n / \|v_n\|_{B_0} \in B_0$ and get

$$\|b_0 w_n\|_B > \eta + d_n \|bw_n\|_{B_1}$$

By the definition of w_n applies:

$$\|b_0 w_n\|_B \leq c \|w_n\|_{B_0} = c$$

and thus it follows $d_n \to \infty (n \to \infty)$, it follows that

$$\|bw_n\|_{B_1} \to 0 \quad (n \to \infty)$$

By construction, $\|w_n\|_{B_0} = 1$ and thus the compact embedding implies $B_0 \overset{b_0}{\hookrightarrow\hookrightarrow} B$, that for a subsequence, which we again denote by (w_n), holds

$$b_0 w_n \to b_0 w \quad \text{in } B \quad (n \to \infty)$$

With the help of the embedding $B \overset{b_1}{\hookrightarrow} B_1$ this results in

$$bw_n = b_1(b_0 w_n) \to b_1(b_0 w) = bw \quad \text{in } B_1 \quad (n \to \infty),$$

which together yields $bw = 0$ in B_1. The injectivity of the embedding b yields $w = 0$ in B_0, from which $b_0 w = 0$ in B. Overall, we therefore have

$$\|b_0 w_n\|_B \to 0 \quad (n \to \infty)$$

which is a contradiction since $\eta > 0$. □

Proof (Aubin-Lions). Let (v_n) be a bounded sequence in W_0. Since W_0 is reflexive, there is a subsequence (v_{n_k}) for which holds:

$$v_{n_k} \rightharpoonup v \text{ in } W_0 \quad (k \to \infty)$$

If we move on to the sequence $u_k := v_{n_k} - v$, we thus obtain

$$u_n \rightharpoonup 0 \text{ in } W_0 \quad (n \to \infty),$$
$$\|u_n\|_{W_0} \leq c \text{ for all } n \in \mathbb{N}.$$

By the Ehrling lemma, for all $\eta > 0$ there is a $d(\eta)$ with

$$\|\boldsymbol{b}_0 u_n\|_{L^{p_0}(I;B)} \leq \eta \|u_n\|_{L^{p_0}(I;B_0)} + d(\eta) \|\boldsymbol{b} u_n\|_{L^{p_0}(I;B_1)},$$

where we use the definition of induced embeddings of Bochner spaces $\boldsymbol{b}_0 : L^{p_0}(I; B_0) \to L^{p_0}(I; B), \boldsymbol{b} : L^{p_0}(I; B_0) \to L^{p_0}(I; B_1)$. Now let $\varepsilon > 0$ be arbitrary. Taking $\eta = \frac{\varepsilon}{2c}$, we obtain

$$\|\boldsymbol{b}_0 u_n\|_{L^{p_0}(I;B)} \leq \frac{\varepsilon}{2} + d(\varepsilon) \|\boldsymbol{b} u_n\|_{L^{p_0}(I;B_1)}.$$

To prove the theorem, it is sufficient to show that

$$\boldsymbol{b} u_n \to 0 \quad \text{in } L^{p_0}(I; B_1) \quad (n \to \infty)$$

We set $p := \min(p_0, p_1)$. It follows that for all $u \in W_0$ $\partial_t(\boldsymbol{b}u) = (\boldsymbol{b}u)'$. Thus $\boldsymbol{b} : W_0 \to W^{1,p}(I; B_1)$ is an embedding. Then $\boldsymbol{b}u$ has a representative $\boldsymbol{b}_c u \in C(\bar{I}; B_1)$. Thus, for all $t \in \bar{I}$ holds

$$\|\boldsymbol{b}_c u_n(t)\|_{B_1} \leq c.$$

For any, but fixed, $\lambda \in (0, 1)$ we define functions $w_n : I \to B_0$, $n \in \mathbb{N}$, by

$$w_n(t) := u_n(\lambda t), \quad \text{for almost all } t \in I,$$

and we obtain

$$\boldsymbol{b}_c w_n(0) = \boldsymbol{b}_c u_n(0)$$
$$\|w_n\|_{L^{p_0}(I;B_0)} = \frac{1}{\lambda^{\frac{1}{p_0}}} \|u_n\|_{L^{p_0}(0,\lambda T;B_0)} \leq c\lambda^{-\frac{1}{p_0}}$$
$$\|(\boldsymbol{b} w_n)'\|_{L^{p_1}(I;B_1)} = \frac{\lambda}{\lambda^{\frac{1}{p_1}}} \|(\boldsymbol{b} u_n)'\|_{L^{p_1}(0,\lambda T;B_1)} \leq c\lambda^{1-\frac{1}{p_1}}.$$

For $\varphi \in C^1(I)$ with $\varphi(T) = 0, \varphi(0) = -1$ applies:

$$\boldsymbol{b}_c w_n(0) = \int_I (\boldsymbol{b} w_n(t)\varphi(t))'\, dt = \int_I \varphi(t)\,(\boldsymbol{b} w_n(t))'\, dt + \int_I \varphi'(t)\boldsymbol{b} w_n(t) dt$$

This yields

$$\begin{aligned}\|\boldsymbol{b}_c w_n(0)\|_{B_1} &\leq c(\varphi)\left\|(\boldsymbol{b} w_n)'\right\|_{L^{p_1}(I;B_1)} + \left\|\int_I \varphi'(t)\boldsymbol{b} w_n(t) dt\right\|_{B_1} \\ &\leq c\lambda^{1-\frac{1}{p_1}} + \left\|\int_I \varphi'(t)\boldsymbol{b} w_n(t) dt\right\|_{B_1}.\end{aligned}$$

Since $p_1 > 1$, we can choose $\lambda \in (0,1)$ such that

$$c\lambda^{1-\frac{1}{p_1}} \leq \varepsilon/2$$

is valid. We have $(w_n) \subseteq L^{p_0}(I; B_0) \hookrightarrow L^1(I; B_0)$ and thus obtain for all $g \in B_0^*$:

$$\begin{aligned}\left\langle g, \int_I w_n(t)\varphi'(t) dt \right\rangle_{B_0} &= \int_I \langle g, w_n(t)\rangle_{B_0} \varphi'(t) dt \\ &= \int_0^{\lambda T} \left\langle g\varphi'\left(\frac{s}{\lambda}\right), u_n(s)\right\rangle_{B_0} ds \to 0 \quad (n \to \infty),\end{aligned}$$

since $\varphi'(\cdot)g \in L^{p_0'}(0, \lambda T; B_0^*)$ and $u_n \to 0$ in $L^{p_0}(0, \lambda T; B_0)\,(n \to \infty)$, due to (2.63). So we have shown that

$$\int_I w_n(t)\varphi'(t) dt \to 0 \text{ in } B_0 \quad (n \to \infty)$$

which due to the compact embedding $B_0 \overset{b_0}{\hookrightarrow} B$ implies

$$\int_I \boldsymbol{b}_0 w_n(t)\varphi'(t) dt \to 0 \text{ in } B \quad (n \to \infty)$$

With $B \overset{b_1}{\hookrightarrow} B_1$, since ε was arbitrary, it yields

$$\boldsymbol{b}_c u_n(0) = \boldsymbol{b}_c w_n(0) \to 0 \text{ in } B_1 \quad (n \to \infty)$$

Now let $s \in I$ be arbitrary. A completely analogous procedure with w_n replaced by

$$\widetilde{w}_n(t) = u_n(s + \lambda t),$$

immediately returns for all $s \in I$

$$\boldsymbol{b}_c u_n(s) \to 0 \text{ in } B_1 \quad (n \to \infty)$$

The majorized convergence theorem applied to the real function sequence $\left(\|\boldsymbol{b}_c u_n(\cdot)\|_{B_1}^{p_0}\right)$, yields the desired result, since $\boldsymbol{b}_c u_n(t) = \boldsymbol{b} u_n(t)$ for almost all $t \in I$. This proves the theorem. □

If we apply the Aubin-Lions theorem to the situation of unsteady p-Laplace problem, we obtain the following result:

Corollary 8.9.1. *Let $p > \frac{2d}{d+2}$, let $\Omega \subseteq \mathbb{R}^d, d \geq 2$, be a bounded domain with boundary $\partial\Omega \in C^{0,1}$ and let B_1 be a reflexive Banach space with $L^2(\Omega) \overset{b_1}{\hookrightarrow} B_1$. Then the space embeds*

$$W_0 := \left\{u \in L^p\left(I; W_0^{1,p}(\Omega)\right) \mid \partial_t(b_1 u) \in L^{p_1}\left(I; B_1\right)\right\}$$

with $1 < p_1 < \infty$, compact to $L^p\left(I; L^q(\Omega)\right)$ if

$$1 \leq q < \frac{pd}{d-p}$$

i.e.

$$W_0 \hookrightarrow\hookrightarrow L^p\left(I; L^q(\Omega)\right)$$

Proof. The embedding $W_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ is compact for $1 \leq q < \frac{pd}{d-p}$. In particular, $q \geq 2$ can be chosen, since $p > \frac{2d}{d+2}$. Thus, the assertion follows immediately from the Aubin-Lions theorem if $q \geq 2$. For $q \in [1, 2)$ we additionally use the embedding $L^2(\Omega) \hookrightarrow L^q(\Omega)$. □

Based on this conclusion, we can prove the existence of a solution to the time-dependent quasilinear PDE. However, further technical complications arise and this procedure has to be repeated for each new problem. Therefore, we proceed differently and develop a general theory without these problems in the next section, which provides an easily applicable abstract existence theorem.

9 Evolution problems with Bochner pseudomonotone operators

We now look again at the initial value problem

$$\begin{aligned} \partial_t(\boldsymbol{e}u) + Au &= b \\ u(0) &= u_0 \end{aligned}$$

We now adapt the notion of pseudomonotoneity to evolution problems. The weak solution to the heat equation lies both in the space $L^2\left(I; W_0^{1,2}(\Omega)\right)$ and in the space $L^\infty\left(I; L^2(\Omega)\right)$. In the case of general Gelfand triples (V, H, j) we define for $p \in (1, \infty)$ and $I = (0, T), T < \infty$,

$$L^p(I; V) \cap_{\boldsymbol{j}} L^\infty(I; H) := \{u \in L^p(I; V) \mid \boldsymbol{j}u \in L^\infty(I; H)\}$$

where $\boldsymbol{j}$ is the induced embedding between Bochner spaces. If we assign the sum norm to $L^p(I; V) \cap_j L^\infty(I; H)$, i.e. $\|\cdot\|_{L^p(I;V)\cap_j L^\infty(I;H)} := \|\cdot\|_{L^p(I;V)} + \|\boldsymbol{j}\cdot\|_{L^\infty(I;H)}$, then $L^p(I; V) \cap_j L^\infty(I; H)$ is a Banach space.

Definition 9.1. *Let (V, H, j) be a Gelfand triple with a reflexive separable Banach space V, and let $p \in (1, \infty)$. We call an operator $\mathcal{A} : L^p(I; V) \cap_{\boldsymbol{j}} L^\infty(I; H) \to (L^p(I; V))^*$ Bochner pseudomonotone, if from*

$$\begin{aligned} u_n &\rightharpoonup u && \text{in } L^p(I; V) \quad (n \to \infty), \\ \boldsymbol{j}u_n &\overset{*}{\rightharpoonup} \boldsymbol{j}u && \text{in } L^\infty(I; H) \quad (n \to \infty), \\ \boldsymbol{j}u_n(t) &\rightharpoonup \boldsymbol{j}u(t) && \text{in } H \text{ for f.a. } t \in I, \quad (n \to \infty), \\ \limsup_{n\to\infty} \langle \mathcal{A}u_n, u_n - u\rangle_{L^p(I;V)} &\leq 0, \end{aligned} \tag{1}$$

it follows that for all $w \in L^p(I;V)$ applies:

$$\langle \mathcal{A}u, u - w\rangle_{L^p(I;V)} \leq \liminf_{n\to\infty} \langle \mathcal{A}u_n, u_n - w\rangle_{L^p(I;V)}$$

The concept of Bochner pseudomonotoneity differs from the pseudomonotoneity by additional requirements on the sequence to be considered. Therefore, every pseudomonotone operator is also Bochner pseudomonotone. On the other hand, there exist Bochner pseudomonotone operators that are not pseudomonotone. The new conditions consider the additional information that can be derived from the time derivation. If the sequence (u_n) comes from a Galerkin system, then the new condition is motivated by the part of the a priori estimate coming from the time derivative. The other condition is motivated by the fact that it can be derived using the partial integration formula for the time derivative and the Galerkin system.
Many properties of Bochner pseudomonotone operators are analogous to the corresponding properties of pseudomonotone operators. Here we only prove the continuity properties that are important for us. Furthermore, analogous to before, we can show that the sum of two Bochner pseudomonotone operators is Bochner pseudomonotone again.

Lemma 9.2. *Let (V, H, j) be a Gelfand triple with a reflexive separable Banach space $V, p \in (1,\infty)$ and $\mathcal{A} : L^p(I;V) \cap_j L^\infty(I;H) \to (L^p(I;V))^*$ a Bochner pseudomonotone operator. Then holds:*

(i) Let $(u_n) \subseteq L^p(I;V) \cap_j L^\infty(I;H)$ satisfy (1)*, and let $(\mathcal{A}u_n)$ be bounded in $(L^p(I;V))^*$. Then the following applies: $\mathcal{A}u_n \rightharpoonup \mathcal{A}u$ in $(L^p(I;V))^*$ $(n\to\infty)$.*

(ii) If $\mathcal{A}$ is additionally locally bounded, then $\mathcal{A}$ is demicontinuous.

Proof. (i) Given a sequence $(u_n) \subseteq L^p(I;V) \cap_j L^\infty(I;H)$ satisfying (1) and for which $(\mathcal{A}u_n)$ is bounded in $(L^p(I;V))^*$. Since $(L^p(I;V))^*$ is reflexive, there is a $b \in (L^p(I;V))^*$ such that for a subsequence: $\mathcal{A}u_{n_k} \rightharpoonup b$ in $(L^p(I;V))^*$ $(k\to\infty)$. This and the Bochner pseudomonotoneity of $\mathcal{A}$ imply for all $w \in L^p(I;V)$

$$\begin{aligned}\langle \mathcal{A}u, u - w\rangle_{L^p(I;V)} \leq& \liminf_{k\to\infty} \langle \mathcal{A}u_{n_k}, u_{n_k} - w\rangle_{L^p(I;V)}\\ \leq& \limsup_{k\to\infty} \langle \mathcal{A}u_{n_k}, u_{n_k} - u\rangle_{L^p(I;V)}\\ &+ \limsup_{k\to\infty} \langle \mathcal{A}u_{n_k}, u - w\rangle_{L^p(I;V)}\\ \leq& \langle b, u - w\rangle_{L^p(I;V)}.\end{aligned}$$

If we replace w by $2u - w$, we get for all $w \in L^p(I;V)$

$$\langle \mathcal{A}u, u - w\rangle_{L^p(I;V)} = \langle b, u - w\rangle_{L^p(I;V)},$$

i.e. $\mathcal{A}u = b$ in $(L^p(I;V))^*$. Since this argument applies to all subsequences of $(\mathcal{A}u_n)$, the convergence principles, see Lemma 0.3 (iv), provides the assertion.
(ii): Let $(u_n) \subseteq L^p(I;V) \cap_j L^\infty(I;H)$ be a sequence such that $u_n \to u$ in $L^p(I;V) \cap_j L^\infty(I;H) (n\to\infty)$. Since $\mathcal{A}$ is locally bounded, the sequence $(\mathcal{A}u_n)$ is also bounded in $(L^p(I;V))^*$. The space $(L^p(I;V))^*$ is reflexive and therefore there is a $b \in (L^p(I;V))^*$ and a subsequence $(\mathcal{A}u_{n_k})$ with $\mathcal{A}u_{n_k} \rightharpoonup b$ in $(L^p(I;V))^*$ $(k\to\infty)$. From this follows $\lim_{k\to\infty} \langle \mathcal{A}u_{n_k}, u_{n_k} - u\rangle_{L^p(I;V)} = 0$. Further follows from $u_{n_k} \to u$ in $L^p(I;V)(k\to\infty)$, it follows that for another subsequence $(u_{n_{k_\ell}})$ and almost all $t \in I$, $u_{n_{k_\ell}}(t) \to u(t)$ in

$V(\ell \to \infty)$ holds. This implies $\boldsymbol{j}u_{n_{k_\ell}}(t) \to \boldsymbol{j}u(t)$ in $H(\ell \to \infty)$ for almost all $t \in I$. The Bochner pseudomonotoneity of $\mathcal{A}$ together with the above convergences implies for all $w \in L^p(I;V)$:

$$\begin{aligned}\langle \mathcal{A}u, u-w\rangle_{L^p(I;V)} &\leq \liminf_{\ell\to\infty} \left\langle \mathcal{A}u_{n_{k_\ell}}, u_{n_{k_\ell}} - w\right\rangle_{L^p(I;V)} \\ &= \langle b, u-w\rangle_{L^p(I;V)}.\end{aligned}$$

Thus, as in (i), $\mathcal{A}u = b$ follows, i.e. $\mathcal{A}u_{n_k} \to \mathcal{A}u$ in $(L^p(I;V))^* \,(k \to \infty)$. The convergence principle Lemma 0.3 (iv) provides, since the above argument applies to arbitrary convergent subsequences,

$$\mathcal{A}u_n \to \mathcal{A}u \quad \text{in } (L^p(I;V))^* \quad (n \to \infty),$$

i.e. $\mathcal{A}$ is demicontinuous. □

We also want to adapt the notion of coercivity to evolution problems and thereby take into account the information coming from the time derivative. This is motivated by the derivation of the a priori estimate.

Definition 9.3. *Let (V,H,j) be a Gelfand triple with a reflexive, separable Banach space V, and let $p \in (1,\infty)$. An operator $\mathcal{A} : L^p(I;V) \cap_j L^\infty(I;H) \to (L^p(I;V))^*$ is called*

(i) Bochner-coercive with respect to $b \in L^{p'}(I;V^*)$ and $u_0 \in H$, if a locally constrained function $M_{\mathcal{A}} : \mathbb{R}_{\geq} \times \mathbb{R}_{\geq} \to \mathbb{R}_{\geq}$ exists, so that for all $u \in L^p(I;V) \cap_j L^\infty(I;H)$ from

$$\frac{1}{2}\|\boldsymbol{j}u(t)\|_H^2 + \left\langle \mathcal{A}u - Rb, u\chi_{[0,t]}\right\rangle_{L^p(I;V)} \leq \frac{1}{2}\|u_0\|_H^2 \qquad \textit{for f.a. } t \in I$$

follows $\|u\|_{L^p(I;V)\cap_j L^\infty(I;H)} \leq M_{\mathcal{A}}\left(\|b\|_{L^{p'}(I;V^*)}, \|u_0\|_H\right).$

(ii) Bochner coercive if $\mathcal{A}$ for all $b \in L^{p'}(I;V^*)$ and $u_0 \in H$ is Bochner coercive with respect to $b \in L^{p'}(I;V^*)$ and $u_0 \in H$ is.

Now we are able to prove an abstract existence theorem with the new terms.

Theorem 9.4. *Let V be a separable reflexive Banach space and (V,H,j) a Gelfand triple, $p \in (1,\infty)$ and $I = (0,T)$ with $0 < T < \infty$. Let $A : V \to V^*$ be an operator such that the induced operator $\mathcal{A} : L^p(I;V) \cap_j L^\infty(I;H) \to (L^p(I;V))^*$ is Bochner-pseudomonotone, Bochner-coercive and bounded. Then for all $u_0 \in H$ and all $b \in L^{p'}(I;V^*)$ there exists a solution $u \in W$ i.e. $u \in W$ satisfies $\boldsymbol{j}u(0) = u_0$ in H and for all $\varphi \in L^p(I;V)$ holds*

$$\int_I \langle \partial_t(\boldsymbol{e}u)(t) + A(u(t)), \varphi(t)\rangle_V \, dt = \int_I \langle b(t), \varphi(t)\rangle_V dt$$

Due to the embedding $W \overset{\boldsymbol{j}}{\hookrightarrow} C(\bar{I};H)$ from before, the solution $u \in W$ has a unique continuous representative in $C(\bar{I};H)$ and thus the initial condition $\boldsymbol{j}u(0) = u_0$ makes sense.

Proof. We prove the theorem using the Galerkin method. Since V is separable, it is easy to consider that there is a sequence $(\tilde{w}_i)_{i\in\mathbb{N}} \subseteq V$ such that for all $n \in \mathbb{N}$ the elements $\{\tilde{w}_i\}_{i=1\ldots n}$ are linearly independent and $\bigcup_{k=1}^\infty \operatorname{span}(\tilde{w}_1,\ldots,\tilde{w}_k)$ is dense in V. Due to the tightness of $j(V)$ in H, the Gram-Schmidt orthonormalization method can be used to construct a sequence $(w_i)_{i\in\mathbb{N}} \subseteq V$ such that $(jw_i)_{i\in\mathbb{N}}$ forms an orthonormal system in H, and $\bigcup_{k=1}^{\infty} \operatorname{span}(jw_1,\ldots,jw_k)$ is dense in H and $\bigcup_{k=1}^{\infty} \operatorname{span}(w_1,\ldots,w_k)$ is dense in V. We set $V_n := \operatorname{span}(w_1,\ldots,w_n)$ and look for approximate solutions $u_n \in C^1\left(\bar{I};V_n\right)$ of the form

$$u_n(t) = \sum_{i=1}^{n} c_n^i(t) w_i$$

which for all $t\in I$ the Galerkin system

$$\frac{d}{dt}\left(j\left(u_n(t)\right), jw_k\right)_H + \left\langle A\left(u_n(t)\right), w_k\right\rangle_V = \left\langle b_n(t), w_k\right\rangle_V, \quad k=1,\ldots,n,$$
$$j\left(u_n(0)\right) = u_0^n$$

can be solved. Here, $u_0^n := \sum_{i=1}^n \left(u_0, jw_i\right)_H jw_i =: \sum_{i=1}^n c_n^{0i} jw_i \in j\left(V_n\right)$ converges strongly in H against $u_0\in H$ and $(b_n)\subseteq C\left(\bar{I};V^*\right)$ is a sequence that converges strongly in $L^{p'}\left(I;V^*\right)$ to $b\in L^{p'}\left(I;V^*\right)$.

(i) Solvability: The Galerkin system is nothing else than the following system of ordinary differential equations for the functions $t\mapsto \mathbf{c}_n(t) = \left(c_n^1(t),\ldots,c_n^n(t)\right)^\top \in\mathbb{R}^n$

$$\frac{d\mathbf{c}_n(t)}{dt} = \mathbf{f}_n\left(t,\mathbf{c}_n(t)\right)$$
$$\mathbf{c}_n(0) = \mathbf{c}_n^0$$

where $f_n^k(t,\mathbf{c}) := \left\langle b_n(t), w_k\right\rangle_V - \left\langle A\left(\sum_{i=1}^n c^i w_i\right), w_k\right\rangle_V, k=1,\ldots,n,$ and $\mathbf{c}_n^0 = \left(c_n^{01},\ldots,c_n^{0n}\right)^\top$. Here we have used the fact that $(jw_i)_{i\in\mathbb{N}}$ is an orthonormal system in H and therefore $\frac{d}{dt}\left(ju_n(t), jw_k\right)_H = \frac{dc_n^k(t)}{dt}$ applies. The operator $\mathcal{A}: L^p(I;V)\cap_j L^\infty(I;H)\to\left(L^p(I;V)\right)^*$ is demicontinuous, since $\mathcal{A}$ is Bochner pseudomonotone and bounded. It follows that the operator $A:V\to V^*$ is also demicontinuous. This and $b_n\in C\left(\bar{I};V^*\right)$ implies that $\mathbf{f}_n:\bar{I}\times\mathbb{R}^n\to\mathbb{R}^n$ is continuous. It follows from Peano's theorem that the system of ordinary differential equations on an interval $[0,\tau^*]$ has a continuously differentiable solution $\mathbf{c}_n$. The standard theory, based on the theorem of Peano, also provides that this solution can be continued, a maximum existence interval $I^* := [0,T^*)$, $T^*\le T$, exists and the continued solution $\mathbf{c}_n$ lies in the space $C^1\left(I^*;\mathbb{R}^n\right)$.

(ii) A priori estimate: To show the global solvability of the Galerkin system, we need a priori estimates. To derive these, we use the Bochner coercivity of $\mathcal{A}: L^p(I;V)\cap_{\boldsymbol{j}} L^\infty(I;H)\to\left(L^p(I;V)\right)^*$. To do this, we form the scalar product of the system with $\mathbf{c}_n(t)$, use $\left(\frac{d\mathbf{c}_n(t)}{dt},\mathbf{c}_n(t)\right)_{\mathbb{R}^n} = \frac{1}{2}\frac{d}{dt}\left|\mathbf{c}_n(t)\right|^2$, integrate over $(0,t)$, use the definitions of u_n and $\mathbf{f}_n$ and note $\left|\mathbf{c}_n(t)\right|^2 = \left\|\boldsymbol{j}u_n(t)\right\|_H^2$ and $\left|\mathbf{c}_n(0)\right|^2 = \left|\mathbf{c}_n^0\right|^2 = \left\|u_0^n\right\|_H^2$, in order for $t\in(0,T^*)$

$$\frac{1}{2}\left\|\boldsymbol{j}u_n(t)\right\|_H^2 + \int_0^t \left\langle A\left(u_n(s)\right) - b_n(s), u_n(s)\right\rangle_V ds = \frac{1}{2}\left\|u_0^n\right\|_H^2 \le \frac{1}{2}\left\|u_0\right\|_H^2$$

can be obtained. With the term $\hat{u}_n := \chi_{[0,T^*)}u_n$, the definition of the induced operator, and the Riesz operator $R: L^{p'}\left(I;V^*\right)\to\left(L^p(I;V)\right)^*$, we can express the last inequality for all $t\in I$ as

$$\frac{1}{2}\left\|\boldsymbol{j}\hat{u}_n(t)\right\|_H^2 + \left\langle \mathcal{A}\hat{u}_n - Rb_n, \hat{u}_n\chi_{[0,t]}\right\rangle_{L^p(I;V)} \le \frac{1}{2}\left\|u_0\right\|_H^2$$

write. The Bochner coercivity thus provides

$$\|\hat{u}_n\|_{L^p(I;V)\cap_j L^\infty(I;H)} \le M_{\mathcal{A}}\left(\|b_n\|_{L^{p'}(I;V^*)}, \|u_0\|_H\right).$$

Since $M_{\mathcal{A}}$ is locally bounded and $b_n \to b$ holds in $L^{p'}(I;V^*)$, we obtain that there is a constant M_0 independent of n such that

$$\sup_{t\in[0,T^*)} \|\boldsymbol{j}u_n(t)\|_H^2 + \int_0^{T^*} \|u_n(s)\|_V^p \, ds \le M_0$$

From $|\mathbf{c}_n(t)|^2 = \|\boldsymbol{j}u_n(t)\|_H^2$ follows $\sup_{t\in[0,T^*)} |\mathbf{c}_n(t)|^2 \le M_0$. Consequently, the limit exists

$$\lim_{t\nearrow T^*} \mathbf{c}_n(t) =: \mathbf{c}_n(T^*)$$

and therefore $\mathbf{c}_n \in C([0,T^*];\mathbb{R}^n) \cap C^1([0,T^*);\mathbb{R}^n)$. If $T^* < T$ we can therefore continue the solution $\mathbf{c}_n$ using Peano's theorem, which is a contradiction to the maximality of the existence interval. Therefore, $T^* = T$ must apply. This yields $\mathbf{c}_n \in C([0,T];\mathbb{R}^n)$, which $u_n \in C([0,T];V_n)$ implies. We have thus proven the global solvability of the Galerkin system on the one hand and the a priori estimate on the other hand

$$\|\boldsymbol{j}u_n\|_{C(\bar{I};H)}^2 + \|u_n\|_{L^p(I;V)}^p \le M_0$$

with a constant M_0 independent of n, is proven. Since the operator $\mathcal{A} : L^p(I;V) \cap_j L^\infty(I;H) \to (L^p(I;V))^*$ is bounded, the second a priori estimate follows from this

$$\|\mathcal{A}u_n\|_{(L^p(I;V))^*} \le c(M_0)$$

with a constant $c(M_0)$ independent of n.

(iii) Convergence of the Galerkin method: Due to the theorems of Riesz, every function $v \in L^\infty(I;H)$ is defined by

$$\langle R\boldsymbol{R}_H v, \varphi\rangle_{L^1(I;H)} := \int_I (v(t),\varphi(t))_H dt = \int_I \langle R_H(v(t)), \varphi(t)\rangle_H \, dt$$

an element $R\boldsymbol{R}_H v \in (L^1(I;H))^*$, where $\boldsymbol{R}_H : L^\infty(I;H) \to L^\infty(I;H^*)$ is the element represented by $R_H : H \to H^*$ is the Riesz isomorphism induced by $R_H : H \to H^*$. It follows that the sequence $(R\boldsymbol{R}_H\boldsymbol{j}u_n)$ is restricted in $(L^1(I;H))^*$. Thus, there thus exists a subsequence, which we again denote by $(R\boldsymbol{R}_H\boldsymbol{j}u_n)$, which converges *-weakly to a $U \in (L^1(I;H))^*$. Since R and $\boldsymbol{R}_H$ are isomorphisms, there exists a function $\hat{u} \in L^\infty(I;H)$ with $R\boldsymbol{R}_H\hat{u} = U$. It is common instead of $R\boldsymbol{R}_H\boldsymbol{j}u_n \overset{*}{\rightharpoonup} R\boldsymbol{R}_H\hat{u}$ in $(L^1(I;H))^*$ to simply write $\boldsymbol{j}u_n \overset{*}{\rightharpoonup} \hat{u}$ in $L^\infty(I;H)$. We will also use this somewhat imprecise expression. It follows that there is a subsequence of (u_n), which we again denote by (u_n), as well as elements $u \in L^p(I;V)$ and $\xi \in L^{p'}(I;V^*)$, so that

$$\begin{aligned}
&u_n \rightharpoonup u \quad \text{in } L^p(I;V) \quad (n\to\infty),\\
&\boldsymbol{j}u_n \overset{*}{\rightharpoonup} \boldsymbol{j}u \quad \text{in } L^\infty(I;H) \quad (n\to\infty),\\
&\mathcal{A}u_n \rightharpoonup R\xi \text{ in } (L^p(I;V))^* \quad (n\to\infty)
\end{aligned}$$

Here we have used that due to the weak continuity of the induced embedding $\boldsymbol{j}$ holds: $\boldsymbol{j}u_n \to \boldsymbol{j}u$ in $L^p(I;H)(n\to\infty)$ and thus the sequence $(\boldsymbol{j}u_n) \subseteq L^\infty(I;H)$ must have the limit value $\boldsymbol{j}u$. We have also used the fact that the limit of the sequence $(\mathcal{A}u_n)$ can be determined using the Riesz operator $R : L^{p'}(I;V^*) \to (L^p(I;V))^*$ from Riesz's representation theorem can be represented by an element $\xi \in L^{p'}(I;V^*)$.

(iv) Next, we want to show further properties of the solution. For all $w \in \bigcup_{k=1}^\infty V_k$ there is a n_0 with $w \in V_{n_0}$. Since u_n is a solution to the Galerkin system, we obtain for all $n \geq n_0$ and all $s \in I$

$$\frac{d}{ds}\left(\boldsymbol{j}u_n(s), jw\right)_H + \langle A\left(u_n(s)\right), w\rangle_V = \langle b_n(s), w\rangle_V$$

where we have used $\boldsymbol{j}u_n(s) = j\left(u_n(s)\right)$. We now multiply equation the variational form of the Galerkin system by a function $\varphi \in C^1(\bar{I})$, integrate with respect to time over $(0,t)$ with an arbitrary, but fixed, $t \in (0,T]$ and, using partial integration in time, the definition of the induced operator $\mathcal{A}$ and the initial condition, obtain

$$\begin{aligned}&-\int_0^t \left(\boldsymbol{j}u_n(s), jw\right)_H \varphi'(s)ds + \left\langle \mathcal{A}u_n, \varphi(\cdot)\chi_{[0,t]}(\cdot)w\right\rangle_{L^p(I;V)}\\ &= \int_0^t \langle b_n(s), w\rangle_V \varphi(s)ds - \left(j\left(u_n(t)\right), jw\right)_H \varphi(t) + \left(u_0^n, jw\right)_H \varphi(0).\end{aligned}$$

For the limit transition we use the fact that from the a priori estimate it follows that for all $t \in (0,T]$ there is a subsequence $\left(\boldsymbol{j}u_{n_k}(t)\right)_{k\in\mathbb{N}}$ that depends on t and a $v_t \in H$ so that

$$\boldsymbol{j}u_{n_k}(t) \to v_t \quad \text{in } H \quad (k\to\infty).$$

Further we use that $\varphi(\cdot)w \in L^p(I;V), \varphi'(\cdot)jw \in L^1(I;H), jw \in H$, as well as $b_n \to b$ in $L^{p'}(I;V^*)\,(n\to\infty)$ and $u_0^n \to u_0$ in $H(n\to\infty)$. Thus it follows for $n = n_k \geq n_0$ the limit transition $(k\to\infty)$ for any, but fixed, $t \in (0,T]$

$$\begin{aligned}&-\int_0^t (\boldsymbol{j}u(s), jw)_H \varphi'(s)ds + \int_0^t \langle \xi(s), w\rangle_V \varphi(s)ds\\ &= \int_0^t \langle b(s), w\rangle_V \varphi(s)ds - \left(v_t, jw\right)_H \varphi(t) + \left(u_0, jw\right)_H \varphi(0).\end{aligned}$$

Since $\bigcup_{k=1}^\infty V_k$ is dense in V, this equation applies to all $w \in V$ and all $\varphi \in C^1(\bar{I})$. If we now choose $t = T$ and $\varphi \in C_0^\infty(I)$, we obtain that

$$\partial_t(\boldsymbol{e}u) = b - \xi \in L^{p'}(I;V^*)$$

We have thus proven $u \in W$. Setting $v(s) := \varphi(s)w$, where $\varphi \in C^1(\bar{I})$ and $w \in V$ and the definition of the generalized time derivative, it follows for any $t \in (0,T]$

$$\begin{aligned}&\int_0^t (\boldsymbol{j}u(s), jw)_H \varphi'(s) + \langle b(s) - \xi(s), w\rangle_V \varphi(s)ds\\ &= \left(\boldsymbol{j}u(t), jw\right)_H \varphi(t) - \left(\boldsymbol{j}u(0), jw\right)_H \varphi(0),\end{aligned}$$

which yields

$$\left(\boldsymbol{j}u(t), jw\right)_H \varphi(t) - \left(\boldsymbol{j}u(0), jw\right)_H \varphi(0) = \left(v_t, jw\right)_H \varphi(t) + \left(u_0, jw\right)_H \varphi(0).$$

If we now choose the function φ for any, but fixed, $t \in (0,T]$ such that $\varphi(t) = 0$ and $\varphi(0) = 1$ or $\varphi(t) = 1$ and $\varphi(0) = 0$, and exploit the fact that $j(V)$ is dense in H, we obtain

$$\boldsymbol{j}u(0) = u_0, \quad \boldsymbol{j}u(t) = v_t, \quad t \in (0,T].$$

We have thus identified the initial condition in particular. It follows $\boldsymbol{j}u_{n_k}(t) \to \boldsymbol{j}u(t)$ in $H(k \to \infty)$. Since this argument applies to all weakly convergent subsequences of $(\boldsymbol{j}u_n(t))$, the convergence principle Lemma 0.3 (iv) yields

$$\boldsymbol{j}u_n(t) \to \boldsymbol{j}u(t) \quad \text{in } H \quad (n \to \infty)$$

for any $t \in (0,T]$. Since $\boldsymbol{j}u(t) = \boldsymbol{j}u(t)$ applies to almost all $t \in I$, we have for almost all $t \in I$

$$\boldsymbol{j}u_n(t) \to \boldsymbol{j}u(t) \quad \text{in } H \quad (n \to \infty)$$

i.e. the crucial new condition in the definition of Bochner pseudomonotoneity is satsifed.

(v) We still have to show $A(u(\cdot)) = \xi(\cdot)$, where we want to exploit the Bochner pseudomonotonicity of $\mathcal{A}$. We have already shown that the sequence (u_n) all besides one condition for the desired property. For this we use $u_n \in C^1\left(\bar{I}; V_n\right)$ to justify the limit transition $t \nearrow T$ and obtain

$$\int_I \langle A\left(u_n(t)\right), u_n(t)\rangle_V \, dt = \int_I \langle b_n(t), u_n(t)\rangle_V \, dt - \frac{1}{2}\left\|j\left(u_n(T)\right)\right\|_H^2 + \frac{1}{2}\left\|u_0^n\right\|_H^2 .$$

From this we deduce with the help of $b_n \to b$ in $L^{p'}\left(I; V^*\right), u_0^n \to u_0$ in $H(n \to \infty)$, the subcontinuity of the norm, as well as the properties of the limit superior that

$$\begin{aligned} \limsup_{n\to\infty} & \int_I \langle A\left(u_n(t)\right), u_n(t)\rangle_V \, dt \\ & \le \int_I \langle b(t), u(t)\rangle_V dt - \frac{1}{2}\left\|\boldsymbol{j}u(T)\right\|_H^2 + \frac{1}{2}\left\|\boldsymbol{j}u(0)\right\|_H^2 \end{aligned}$$

is valid. On the other hand, it follows

$$\begin{aligned} -\frac{1}{2}\left\|\boldsymbol{j}u(T)\right\|_H^2 + \frac{1}{2}\left\|\boldsymbol{j}u(0)\right\|_H^2 &= -\int_I \langle \partial_t(\boldsymbol{e}u)(t), u(t)\rangle_V \, dt \\ &= \int_I \langle \xi(t) - b(t), u(t)\rangle_V dt \end{aligned}$$

which yields

$$\limsup_{n\to\infty} \int_I \langle A\left(u_n(t)\right), u_n(t)\rangle_V \, dt \le \int_I \langle \xi(t), u(t)\rangle_V dt.$$

If we use the definition of the induced operator $\mathcal{A}$ and the Riesz operator R in this inequality, we obtain

$$\limsup_{n\to\infty} \langle \mathcal{A}u_n, u_n - u\rangle_{L^p(I;V)} \le 0$$

i.e. the sequence (u_n) satisfies the desired property in the definition of Bochner pseudomonotonicity. Since the operator $\mathcal{A}$ is Bochner pseudomonotone and the sequence (u_n) satisfies all the conditions, it yields that $\mathcal{A}u_n \to \mathcal{A}u$ converges to $(L^p(I;V))^*\,(n \to \infty)$. We get sich $\mathcal{A}u = R\xi$ in $(L^p(I;V))^*$, which is equivalent to $A(u(\cdot)) = \xi(\cdot)$ in $L^{p'}\left(I; V^*\right)$. This implies that $u \in W$ is a solution i.e. the weak formulation in the theorem is fulfilled. Thus, the theorem is completely proven.

□